Regularization of inverse problems in image processing

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Régularisation de problèmes inverses pour le traitement des images

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CHAPITRE 1

Introduction

1.1 Introduction générale

Les problèmes inverses consistent à retrouver une donnée qui a été transformée ou altérée. L'élaboration de méthodes de restauration efficaces trouve des applications dans des domaines aussi variés que la reconnaissance vocale, l'imagerie médicale et spatiale, la géophysique, la finance pour ne citer que quelques exemples.

Dans le domaine de l'imagerie, l'acquisition, la transmission et le stockage sont des étapes qui altèrent les données. Une image idéale peut ainsi à chacune de ces étapes subir l'addition d'un flou, d'un bruit ou des dégradations plus sévères. L'objectif du traitement d'images consiste soit à reconstruire cette image, soit à la simplifier pour en extraire une information pertinente.

La résolution de ce type de problèmes met en oeuvre de nombreux outils mathématiques comme les équations aux dérivées partielles, des méthodes statistiques, l'analyse harmonique, l'analyse numérique, le calcul des variations.

Cette thèse s'inscrit dans le cadre des approches variationnelles. Dans ce type de méthodes, la donnée restaurée est vue comme un minimiseur d'une énergie qui reflète une connaissance *a priori* de la structure du problème. Dans le cadre qui nous intéresse, l'énergie en question doit rendre compte des propriétés intrinsèques des images pour espérer avoir une bonne restauration. Le modèle doit également prendre en considération le procédé de dégradation. La grande variété des images et des perturbations rend ce problème difficile et donc intéressant.

La thèse mêle modélisation, étude théorique et algorithmique accompagnées de résultats numériques. Elle est organisée en quatre parties :

- Dans le Chapitre 2, nous nous intéressons à une étude théorique des propriétés de la pénalisation par variation totale. Pour la restauration des images, la variation totale, en tant qu'outil de régularisation, a l'avantage de préserver les discontinuités tout en créant des zones lisses. Nous établissons pour des énergies anisotropes générales les résultats connus pour les discontinuités de minimiseurs de la variation totale. Cette étude nous amène notamment à nous interroger sur la régularité des ensembles de niveau et à faire appel à la théorie de régularité des surfaces minimales. Nous montrons également qu'il est inévitable que l'image restaurée contienne des grandes zones constantes, phénomène indésirable que l'on appelle *staircasing*. Enfin, nous établissons un lien intéressant avec le flot de la variation totale pour une donnée radiale. Cette relation nous permet de raffiner les résultats précédents dans le contexte des fonctions radiales.

- Dans le Chapitre 3, nous proposons une alternative à la variation totale qui diminue dans certains cas le phénomène de staircasing. L'étude de cette fonctionnelle nous amène à établir une formulation duale, utile pour démontrer que la variante proposée coïncide avec la variation totale sur les images "cartoon". Une comparaison numérique avec la variation totale est proposée.

- Ces dernières années, les méthodes non-locales exploitant les auto-similarités dans les images ont connu un vif succès. Dans le Chapitre 4, nous adaptons ce type d'approche au problème de restauration de spectre de Fourier. Nous présentons en outre des applications pour des problèmes inverses généraux. Cette partie, essentiellement numérique a nécessité la rédaction d'un programme en Matlab, mais également en C, pour paralléliser le calcul.

- Le dernier chapitre est consacré aux problèmes algorithmiques inhérents à l'optimisation des énergies convexes considérées dans la thèse. Nous étudions notamment la convergence et la complexité d'une famille d'algorithmes dits Primal-Dual récemment développée.

Avant de poursuivre nous allons faire quelques rappels sur les différents modèles variationnels qui dominent la littérature contemporaine.

1.2 Les méthodes variationnelles en traitement des images

1.2.1 Les problèmes inverses nécessitent une régularisation

Une image numérique est en général une matrice $g = (g_{i,j})_{i,j}$ où $g_{i,j} \in \{0, \cdots, 255\}$ pour les images en niveau de gris sur 8 bits et $g_{i,j} \in \{0, \cdots, 255\}^3$ pour les images couleur en représentation RGB sur 8 bits. Il existe des représentations avec d'avantage de niveaux (typiquement 16 bits) en imagerie médicale et en imagerie satellitaire par exemple. Un élément de la matrice g est appelé *pixel*.

Prenons l'exemple d'une image capturée par un appareil photo numérique. Elle est le résultat d'un échantillonage spatial puisque la valeur d'un pixel est la moyenne de l'intensité lumineuse reçue par un capteur et d'une quantification qui consiste à se ramener à un nombre de niveaux de gris fini. On passe ainsi d'une donnée continue à une donnée discrète qui donne lieu à un traitement informatisé.

Le traitement des images a ainsi pour objet :

- la restauration (débruitage, défloutage, reconstruction de zones manquantes : problème d'*inpainting*, etc.),
- l'interprétation (extraction d'éléments caractéristiques par segmentation, classification, comparaison d'images, etc.),
- la compression (réduction de l'information nécessaire pour avoir une bonne représentation de l'image).

Ces traitements sont dans la pratique effectués dans un domaine discret. Néanmoins, il est souvent intéressant de les définir et de les analyser dans un cadre continu. Le lien entre les deux approches est obtenu par des résultats d'approximation par Γ-convergence.

Ainsi, dans la suite, une image g_0 idéale sera vue comme une fonction d'un domaine ouvert $\Omega \subset \mathbb{R}^N \to \mathbb{R}$, de carré intégrable. Un modèle rudimentaire consiste à voir l'image comme une juxtaposition d'objets. Cela revient à supposer que g_0 est régulière par morceaux en dehors d'un ensemble de discontinuité lui-même régulier. Cependant nous verrons dans la suite que ce modèle est loin d'être parfait. En ce qui concerne l'image transformée g, elle peut être vue comme le résultat d'une transformation linéaire A suivie de l'addition d'un bruit blanc Gaussien n de variance σ ; en d'autres termes $g = Ag_0 + n$. D'autres dégradations sont également envisagées dans la littérature (bruit multiplicatif, bruit Poissonnien).

Un candidat u pour la restauration doit rester proche de g après application de la transformation A, tout en minimisant une contrainte de régularité, quantifiée par un opérateur $\mathcal{R}$. Dans la pratique, cela consiste à chercher l'image restaurée $\bar{u}$ comme minimiseur d'un problème non contraint

$$\min_u \frac{1}{2}\|Au - g\|_2^2 + \lambda \mathcal{R}(u) \tag{1.2.1}$$

dit problème régularisé.

Une autre approche qui en général n'est pas équivalente consiste à résoudre le problème contraint

$$\min\{\mathcal{R}(u),\ u \text{ tel que } \|Au - g\|_2 = \sigma\}. \tag{1.2.2}$$

Les méthodes par régularisation développées pour le traitement des images peuvent évidemment trouver des applications pour les signaux 1D comme le montre l'exemple suivant :

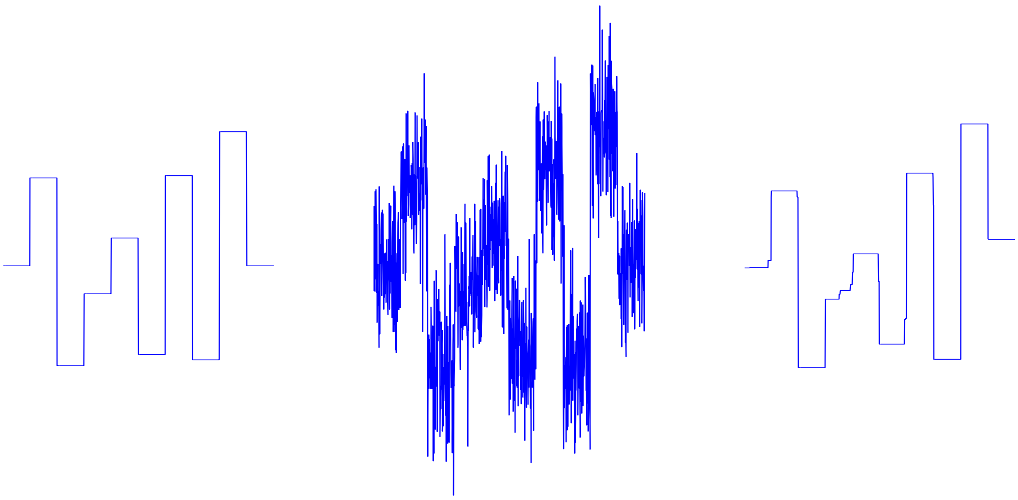


Fig. 1.1: De gauche à droite, le signal constant par morceaux originel, le signal corrompu et le signal restauré par pénalisation de la variation totale (cf. ci-après).

1.2.2 L'interprétation bayésienne

Supposons que Ω est fini et qu'une image naturelle est la réalisation d'un processus aléatoire qui suit la distribution de Gibbs

$$p(u) = \frac{1}{Z(\beta)} \exp(-\beta \mathcal{R}(u)).$$

Ici, $Z(\beta)$ est un facteur de renormalisation. En d'autres termes, les images irrégulières, correspondant à des grandes valeurs de $\mathcal{R}(u)$, sont peu probables.

Grâce à la loi de Bayes

$$p(u|g)p(g) = p(g|u)p(u),$$

on peut calculer la probabilité *a posteriori* de u étant donné g. Comme nous avons supposé que le bruit était blanc Gaussien de variance σ,

$$p(g|u) = \frac{1}{Z(g)} \exp\left(-\frac{\|Au - g\|_2^2}{2\sigma^2}\right).$$

En somme,

$$p(u|g) = C(g, \beta) \exp\left(-\frac{\|Au - g\|_2^2}{2\sigma^2} - \beta\mathcal{R}(u)\right).$$

Une première idée, consistant à maximiser cette probabilité, redonne exactement le problème (1.2.1) pour le choix $\lambda = 2\sigma^2\beta$. Une autre approche envisageable revient à calculer l'espérance

$$\mathbb{E}(u|g) = \int_{\mathbb{R}^\Omega} up(u|g)du,$$

qui est donc une image. Cette approche peut être interprétée comme une modification de risque.

Pour avoir une restauration acceptable, trois pistes s'offrent à nous :

- Trouver le paramètre optimal λ. Dans la pratique, ce paramètre est laissé libre et permet à un utilisateur final de choisir le niveau de régularisation souhaité.
- Envisager d'autres types de risques. L'emploi d'un risque quadratique en traitement des images a été étudié par Louchet et Moisan dans [109].
- Choisir une régularisation $\mathcal{R}$, ce qui revient à se donner une distribution sur les images naturelles.

Le choix de la régularisation $\mathcal{R}$ est difficile mais crucial et a fait l'objet d'une littérature abondante dans le domaine des problèmes inverses. Nous allons, dans la suite de cette introduction, suivre les étapes qui ont mené au développement de méthodes efficaces.

1.2.3 L'approche de Tychonov

Une approche populaire, initiée par Tychonov en 1963, consiste à pénaliser les grandes oscillations du bruit par adjonction dans l'énergie de la norme L^2 du gradient. L'image restaurée $\bar{u}$ est obtenue comme solution du problème convexe

$$\min_u \frac{1}{2}\|Au - g\|_2^2 + \frac{\lambda}{2}\int_\Omega |\nabla u|^2,$$

où la minimisation se fait dans l'espace de Sobolev $H^1(\Omega)$.

Remarquons que l'interprétation bayésienne et le changement de risque n'apportent rien dans ce cas puisque l'énergie considérée est quadratique et donc $\mathbb{E}(u|g) = \bar{u}$.

Cette approche présente un avantage important d'un point de vue numérique. L'énergie se minimise aisément grâce à une méthode de type gradient conjugué et ce en un nombre fini d'itérations. On peut également envisager de résoudre l'équation d'Euler-Lagrange associée

$$A^*(Au - g) - \lambda\Delta u = 0 \tag{1.2.3}$$

du côté Fourier. Néanmoins, ce modèle présente un inconvénient qui est reflété par le comportement diffusif de l'équation (1.2.3) : le minimiseur ne peut contenir de discontinuités le long d'hypersurfaces. Autrement dit, ce modèle produit des images restaurées où les zones de transition ne sont pas clairement marquées.

Plus généralement, il a été envisagé d'employer la norme L^p, pour $p > 1$, avec à la clé le même type de phénomène. L'équation d'Euler-Lagrange s'écrit alors

$$A^*(Au - g) - \lambda\Delta_p u = 0$$

où $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ est le p-Laplacien.

1.2.4 Une régularisation non-lisse pour préserver les discontinuités

La parution de l'article fondateur de Rudin, Osher et Fatemi [141] (ROF dans la suite) a marqué l'avènement des méthodes non linéaires et non lisses dans le traitement des images. Pour éviter le comportement diffusif décrit ci-dessus, leur idée est de pénaliser la norme L^1 du gradient. Le problème de minimisation dans l'espace $W^{1,1}(\Omega)$ est mal posé et la solution doit être cherchée dans son complété pour la norme $\|\cdot\|_{W^{1,1}}$: l'espace des fonctions à variation bornée $BV(\Omega)$. Cet espace peut être vu comme l'ensemble des fonctions $u \in L^1_{loc}(\Omega)$ dont la dérivée au sens des distributions Du est une mesure de Radon bornée. L'image restaurée $\bar{u}$ est alors obtenue comme minimiseur du problème

$$\min_{u\in BV(\Omega)} \frac{1}{2}\|Au - g\|_2^2 + \lambda\int_\Omega |Du|, \tag{1.2.4}$$

où on a pénalisé la variation totale de la dérivée Du, dorénavant notée $TV(u)$. L'espace $BV(\Omega)$ jouit de bonnes propriétés de semi-continuité et de compacité ce qui garantit l'existence d'une solution au problème (1.2.4).

Chambolle et Lions [57] démontrent que, moyennant un bruit dont la variance n'est pas trop élevée ($\sigma \leq \left\| g - \int_\Omega g \right\|_2$), il existe un paramètre λ pour lequel il est équivalent de résoudre le problème avec contrainte (1.2.2).

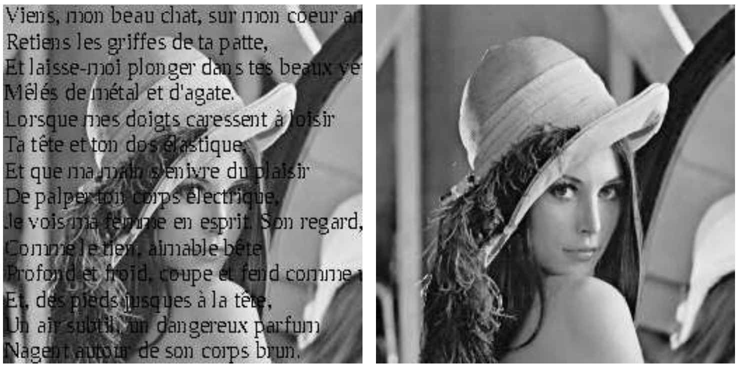


FIG. 1.2: A gauche l'image corrompue et à droite, un exemple de restauration par minimisation de la variation totale.

Différentes modifications du modèle ont été envisagées dans la littérature. Nikolova propose dans [134] d'utiliser une norme L^1 dans l'attache à la donnée classique. Cela donne de meilleurs résultats en présence de bruit impulsionnel. L'idée d'utiliser une attache à la donnée non-différentiable en traitement du signal remonte à [4]. Gilboa et al. [91], envisagent un paramètre de régularisation qui s'adapte aux zones de l'image à restaurer. Le paramètre λ devient alors une fonction de l'espace et doit être interprété comme un poids. D'autres variantes consistent à considérer des versions différentiables de la semi-norme TV (Huber, TV_ε). L'implémentation numérique du problème de minimisation en est alors facilitée

De plus, il est bien connu que la minimisation de la variation totale favorise l'apparition de régions constantes dans l'image restaurée. Ce phénomène, qualifié de *staircasing* dans la littérature en anglais, n'est pas toujours souhaitable. Considérer des variantes différentiables de la variation totale tend alors à diminuer le phénomène de staircasing comme l'a démontré Nikolova [133] dans un cadre discret.

Le phénomène de staircasing en lui-même a été mis en évidence en dimension $N = 1$ dans [140, 43] et Nikolova [133] l'a étudié pour des énergies discrétisées en dimension supérieure.

Une approche similaire à la minimisation de la variation totale consiste à suivre le flot de la variation totale pour obtenir une image débruitée. Plus précisément, étant donnés un ouvert $\Omega \subset \mathbb{R}^N$ et une condition initiale $u(0) = g \in L^2(\Omega)$, on obtient une famille d'images restaurées comme unique solution du flot de gradient :

$$-\partial_t u(t) \in \partial TV(u(t)) \text{ presque tout } t \in [0,T]. \tag{1.2.5}$$

L'étude des propriétés de ce flot est condensé dans le livre [13].

Au final, le modèle basé sur la variation totale et plus généralement les fonctions à variation bornée ont trouvé des applications bien au-delà des problèmes qui se posent en traitement d'images.

1.2.5 Lien avec les surfaces minimales

Un autre aspect fondamental des fonctions de classe BV est le lien avec le problème de surface minimale. En effet, la formule de la coaire permet de montrer que les surniveaux $E_t = \{\bar{u} > t\}$ du minimiseur $\bar{u}$ de (1.2.4) résolvent

$$\min_E \lambda P(E) + \int_E t - g. \tag{1.2.6}$$

Ici, $P(E) = \mathcal{H}^{N-1}(\partial E)$ désigne le périmètre de l'ensemble E et le second terme doit être interprété comme une contrainte sur la courbure moyenne de la surface minimisante E_t. Le lien entre la minimisation de la variation totale et le problème de surface minimale était déjà connu par De Giorgi dans les années 1950.

L'étude de ce type de problèmes est l'objet de la théorie géométrique de la mesure. Le problème de Plateau en est un autre exemple et consiste à montrer qu'il existe une surface minimale avec un bord imposé. Ce problème porte le nom de Joseph Plateau qui s'est intéressé au XIX^{e} siècle à la formation des films de savon.

Fig. 1.3: Deux surfaces minimales qui résolvent le problème de Plateau.

Une question fondamentale a consisté à se demander s'il existe, pour ce type de problèmes, des solutions régulières en un sens classique. Les travaux de De Giorgi, Bombieri, Massari, Tamanini et Federer [31, 148, 115, 116, 117, 85] ont permis de répondre par l'affirmative à cette question en dimension $N < 8$ et ce dans divers contextes. En 1969, Bombieri, De Giorgi et Giusti [32] donnent un exemple de cône minimal pour le problème de Plateau en dimension $N = 8$, exhibant ainsi une surface minimale comportant une singularité. Enfin, en dimension supérieure, Federer démontre en 1970 que l'ensemble de singularité a une dimension de Hausdorff inférieure à $N - 8$.

Pour une introduction pédagogique aux questions de régularité des minimiseurs du problème (1.2.6), on pourra consulter les notes de cours d'Ambrosio [7] ou le livre récent de Maggi [111, Chapitre 3].

Le lien entre l'énergie (1.2.4) et le problème de surface minimale sous-jacent est notamment exploité dans [48, 49]. Dans ces travaux, la proximité des ensembles de niveau permettent de déduire des propriétés en ce qui concerne les discontinuités et la régularité du minimiseur du problème (1.2.4). Dans leur étude, la régularité des ensembles de niveau joue un rôle fondamental.

Pour les minimiseurs de (1.2.4), des résultats de ce type étaient partiellement connus grâce au calcul de solutions explicites. Mentionnons notamment les travaux [13, 2] qui s'intéressent à une donnée qui est la caractéristique d'un convexe ou la somme de caractéristiques d'ensembles convexes éloignés. Les résultats de ces papiers sont complétés par les articles récents [3, 51] où les auteurs s'intéressent à une donnée dans le plan qui est l'union de deux disques ou deux carrés proches.

1.2.6 Un exemple de problème aux frontières libres

La littérature regorge de modèles concurrents de la variation totale. Faute d'être exhaustif, nous ne pouvons pas oublier l'approche de Mumford et Shah [128] qui fait partie de la grande famille des problèmes aux frontières libres.

Ce modèle a l'avantage de concilier l'approche de Tychonov et celle de Rudin, Osher et Fatemi et revient à chercher un couple $(\bar{u}, \bar{K})$ qui minimise

$$\min_{u,K} \frac{1}{2}\|u - g\|_2^2 + \frac{\lambda}{2} \int_{\Omega \setminus K} |\nabla u|^2 + \mu \mathcal{H}^{N-1}(K), \tag{1.2.7}$$

parmi les ensembles K fermés dans Ω et $u \in H^1(\Omega \setminus K)$.

La question de l'existence d'un couple solution est loin d'être triviale. Une preuve d'existence utilise la formulation faible

$$\min_{u \in SBV(\Omega)} \frac{1}{2}\|u - g\|_2^2 + \frac{\lambda}{2} \int_{\Omega \setminus S_u} |\nabla u|^2 + \mu \mathcal{H}^{N-1}(S_u),$$

où l'image restaurée $\bar{u}$ est recherchée dans la sous-classe $SBV \subset BV$ et $\bar{K} = \overline{S_{\bar{u}}}$, l'ensemble singulier de $\bar{u}$. Cette démonstration repose sur une propriété fondamentale de compacité de l'espace $SBV(\Omega)$ (nous renvoyons pour plus de détails à [8]).

Une mise en oeuvre numérique se base sur le résultat de Γ-convergence d'Ambrosio et Tortorelli [12]. Ainsi, étant donnée une approximation η_ε de K, ils établissent la Γ-convergence [38] de l'énergie de champ de phase

$$\frac{\lambda}{2}\int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 + \mu \int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon} \right)$$

vers le terme de régularisation de l'énergie de Mumford et Shah (1.2.7). Les méthodes de champ de phase sont également redoutables pour approcher le périmètre [123] et simuler le mouvement par courbure moyenne via l'approximation d'Allen-Cahn [39]. Dans [93, 52], Chambolle et Gobbino proposent une approximation par différences finies inspirée d'une conjecture de De Giorgi. Enfin, Chan et Vese [64] envisagent une méthode de type level set pour le problème de segmentation, correspondant au cas où le paramètre $\lambda \to +\infty$.

L'approche de Mumford et Shah soulève encore de nombreuses questions palpitantes mais difficiles. En dimension $N = 2$, Mumford et Shah ont notamment conjecturé que la segmentation minimale $\bar{K}$ est une union finie d'arcs lisses et que ceux-ci se croisent toujours par nombre de trois en formant des angles de $\frac{2\pi}{3}$.

Dans [87], Fornasier, March et Solombrino ont étendu l'étude variationnelle au cas où l'attache à la donnée tient compte d'une perturbation linéaire A de la donnée.

1.2.7 Régularisations d'ordre supérieur

La pénalisation des dérivées d'ordre supérieur apparaît dans la littérature comme une solution pour éviter le phénomène de staircasing observé avec la variation totale. Dans [57], la fonctionnelle

$$\min_{\substack{u_1 \in BV(\Omega) \\ u_2 \in BH(\Omega)}} \frac{1}{2}\|A(u_1+u_2) - g\|_2^2 + \lambda \int_\Omega |Du_1| + \mu \int_\Omega |D(D(u_2))| \tag{1.2.8}$$

permet d'atteindre cet objectif. Remarquons que $BH(\Omega)$ est l'espace des fonctions dites à Hessiennes bornées étudié par Demengel dans [76]. Une idée similaire, développée dans [61], revient à remplacer la dernière intégrale dans (1.2.8) par la norme L^1 du Laplacien. Les articles [26, 40] utilisent une variante de la définition duale de la variation totale qui fait également intervenir des dérivées d'ordre supérieur de u. Dans [28], les auteurs considèrent une modification de l'équation de Cahn-Hilliard où apparaît le terme $\Delta(\Delta u)$. Cependant, il n'est pas clair que le flot considéré dérive d'une énergie.

Pour des raisons différentes, Ambrosio et Masnou considèrent dans [10] une énergie de Willmore

$$\mathcal{R}(u) = \int_\Omega |Du| \left(1 + \left|\operatorname{div}\left(\frac{Du}{|Du|}\right)\right|^p\right).$$

Une telle régularisation permet non seulement de tenir compte du périmètre des ensembles de niveau mais également de leur courbure moyenne.

Fig. 1.4: A gauche, la zone en noir doit être recouvrée, restauration par minimisation du périmètre et restauration avec prise en compte de la courbure respectivement.

D'autres travaux récents intègrent également une énergie de courbure de ce type (cf. par exemple Boscain et al. [34]). L'intégration d'un terme de courbure à la fonctionnelle de Mumford-Shah a été envisagée par Esedoglu et Shen dans [81] et les propriétés variationnelles de cette nouvelle fonctionnelle sont étudiées dans [23] par Bellettini et March.

1.2.8 Prise en compte de la texture

Les modèles basés sur la pénalisation des dérivées connaissent cependant un défaut important. En effet, ils sont connus pour dégrader les textures puisque celles-ci présentent un gradient élevé. D'autre part, des méthodes statistiques (voir [95]) mettent en évidence le fait que les images ne sont pas des fonctions de classe BV. Ceci a motivé la recherche de modèles adaptés aux images naturelles.

Pour pallier les faiblesses des modèles précédents, une première méthode consiste à décomposer l'image en une composante géométrique représentant la structure des éléments de l'image et une seconde composante contenant les textures. Ces deux parties peuvent ensuite être traitées de manière indépendante.

Yves Meyer a initié ce type d'approches dans [120] en remplaçant la norme L^2 dans l'énergie de ROF par une norme $\|\cdot\|_G$. L'espace de Banach G associé à cette norme contient des signaux présentant de grandes oscillations et en particulier le bruit et les textures. D'autres auteurs ont poussé plus loin cette idée en considérant des espaces fonctionnels de plus en plus sophistiqués (Sobolev, BMO, H^{-1}, Hilbert à noyau). Mentionnons en particulier le travail [17] où les auteurs emploient l'espace de Besov homogène $E = \dot{B}^{\infty}_{-1,\infty}$ pour capturer le bruit blanc Gaussien. Ils parviennent ainsi à décomposer une image bruitée en trois composantes cartoon u, texture v et bruit n par minimisation de l'énergie

$$\min_{\substack{u\in BV(\Omega)\\ \|v\|_G\leq\mu,\ \|n\|_E\leq\nu}} \frac{1}{2}\|g-(u+v+n)\|_2^2 + \lambda\int_\Omega |Du|.$$

La décomposition structure-texture trouve notamment une application intéressante dans [27] où les auteurs décomposent l'image pour le problème d'inpainting. Ainsi, la texture est retrouvée par copier-coller alors que le contenu géométrique est restauré par un procédé de diffusion. Cette idée est également exploitée dans [119] pour une donnée présentant des textures dites "localement parallèles".

1.2.9 L'ère des méthodes non-locales

Ces dernières années, les méthodes non-locales ont remporté tous les suffrages en se plaçant en tête des tests comparatifs. L'idée simple mais fondamentale repose sur un axiome de base : les images présentent des régions similaires qui peuvent être distantes les unes des autres.

Buades, Coll et Morel [44] exploitent cette idée pour le problème de débruitage en proposant le filtre à moyennes non-locales (NLMeans). Celui-ci consiste à moyenner entre elles les portions de l'image (*patches* en anglais) qui sont semblables. L'idée de comparer les niveaux de gris des pixels remonte en fait à [157].

Les articles [91, 90, 135, 15] permettent d'exploiter cette propriété de régularité des images pour des problèmes inverses généraux en proposant de minimiser une énergie du type

$$\min_u \frac{1}{2}\|Au-g\|_2^2 + \lambda\int_{\Omega\times\Omega} \|p_u(x)-p_u(y)\| \exp\left(-\frac{d(p_g(x),p_g(y))}{h}\right) dxdy,$$

où $p_u(x)$ désigne un patch de l'image u centré en x. De plus, $d(p_g(x),p_g(y)) = \|p_g(x)-p_g(y)\|_2$ quantifie la proximité entre deux patches de l'image corrompue.

L'usage des méthodes non-locales a largement été adopté par la communauté de traitement d'images et on connaît désormais de nombreuses variations autour de ce thème. Mentionnons notamment le filtre BM3D [70] qui se hisse tout en haut de l'état de l'art pour le problème de débruitage.

1.2.10 Les représentations parcimonieuses

L'idée d'utiliser une base d'ondelettes a été envisagée par Grossman et Morlet dans les années 80 comme une alternative à la base de Fourier. Les ondelettes ont été démocratisées via l'approche multirésolution de Mallat [113] et sont utilisées au quotidien pour la compression des images grâce au standard JPEG2000. L'approche par ondelettes s'appuie sur le fait que les images (continues) sont invariantes par changement d'échelle. Les ondelettes trouvent des applications intéressantes pour l'ensemble des problèmes inverses et en particulier pour le problème de débruitage. Dans ce dernier cas, il est possible d'obtenir une image débruitée par approximation non linéaire. En d'autres termes, on ne conserve que les coefficients d'ondelettes de l'image g qui sont supérieurs à un seuil λ (*soft thresholding*). Or, une estimation sur la taille des coefficients d'ondelettes caractérise l'appartenance à un espace de Besov. Ce procédé revient donc à minimiser

$$\min_{u \in \dot{B}^1_{1,1}} \frac{1}{2}\|Au - g\|_2^2 + \lambda \|u\|_{\dot{B}^1_{1,1}},$$

où $\dot{B}^1_{1,1}$ est l'espace de Besov homogène muni de sa norme $\|\cdot\|_{\dot{B}^1_{1,1}}$ et $A = Id$.

Cette dernière décennie a vu apparaître de nouvelles familles d'ondelettes dites géométriques (curvelets [47], bandlets [107, 114],...). Celles-ci prennent mieux en compte la géométrie des discontinuités et permettent d'obtenir une bonne représentation avec très peu de coefficients, représentation dite parcimonieuse (*sparse* en anglais).

Une catégorie de méthodes plus générale est basée sur la construction d'un dictionnaire D formé d'une famille de vecteurs (qui ne sont pas nécessairement des ondelettes). Dans ce cas, l'image restaurée $\bar{u}$ s'exprime de la manière la plus simple possible comme combinaison des vecteurs de D. En d'autres termes, on cherche un vecteur de coefficients $\bar{x}$ qui contienne un maximum de zéros telle que $\bar{u} = D\bar{x}$. Évidemment la solution doit rester proche de la donnée initiale ce qui revient à chercher les coefficients $\bar{x}$ comme solution du problème

$$\min_x \frac{1}{2}\|ADx - g\|_2^2 + \lambda\|x\|_1. \tag{1.2.9}$$

Remarquons que le dictionnaire engendre l'espace sans forcément constituer une base et généralise en ce sens l'approche par ondelettes. Le choix de la norme L^1 est cruciale puisqu'elle impose dans la pratique une représentation parcimonieuse [78].

Certaines publications récentes visent à apprendre le meilleur dictionnaire, soit en utilisant des bases de données d'images similaires [1] (méthodes dites *offline*), soit à partir de l'image à restaurer directement (en *online*). Dans ce dernier cas, il est possible d'utiliser un dictionnaire adapté aux patches de l'image [112], ce qui fait le lien avec les méthodes non-locales.

1.3 Contributions de la thèse

1.3.1 Propriétés fines des minimiseurs de la variation totale

L'étude des propriétés des minimiseurs de la variation totale est au coeur du Chapitre 2 de cette thèse. Nous nous intéresserons notamment au comportement des discontinuités et des zones homogènes des minimiseurs de ce type de problème. Nos résultats ont donné lieu à trois publications [98, 99, 50].

Les discontinuités du minimiseur

Notre travail a consisté, dans un premier temps, à généraliser le résultat d'inclusion des discontinuités du minimiseur de ROF [48] pour des énergies générales du type

$$\mathcal{E}(u) = \int_\Omega \Phi(x, Du)dx + \int_\Omega \Psi(x, u(x))dx. \tag{1.3.1}$$

Ici, Φ est une anisotropie dérivant d'une métrique de Finsler, essentiellement lisse en dehors de $\Omega \times \mathbb{R}^N \setminus \{0\}$, positivement 1-homogène et elliptique en la seconde variable tandis que Ψ est mesurable en la première variable, strictement convexe et coercive en la seconde.

Dans ce contexte, le résultat de [48] devient :

Théorème 1.3.1. *Soient Φ et Ψ comme ci-dessus. Supposons que pour un ensemble dénombrable D dense dans $\mathbb{R}$, on ait*

$$\partial_t \Psi(\cdot, t) \in BV(\Omega) \cap L^\infty(\Omega), \ \forall t \in D.$$

Alors, si u est le minimiseur de $\mathcal{E}$, son ensemble de discontinuité satisfait

$$J_u \subset \bigcup_{t \in D} J_{\partial_t \Psi(\cdot, t)}$$

à un ensemble de mesure $\mathcal{H}^{N-1}$ négligeable près.

L'idée de la preuve, dans le cas plus simple de la fonctionnelle de ROF, exploite le lien avec le problème de minimisation du périmètre (1.2.6). Formellement, l'ensemble de surniveau $E_t = \{u > t\}$ satisfait l'équation d'Euler-Lagrange

$$\kappa_{E_t} + \frac{1}{\lambda}(t - g) = 0 \text{ sur } \partial E_t, \tag{1.3.2}$$

où κ_{E_t} désigne la courbure moyenne du bord ∂E_t. Ainsi, si l'on suppose qu'il existe $x \in J_u \setminus J_g$, alors on peut trouver deux valeurs $t_1 < t_2$ telles que $x \in \partial E_{t_1} \cap \partial E_{t_2} \setminus J_g$. Or $E_{t_2} \subset E_{t_1}$ et ces deux ensembles se touchent en x.

On en déduit que l'ensemble de niveau le plus petit est le plus élevé mais a la courbure moyenne la plus petite. Ceci contredit le fait que x est un point de contact des deux niveaux.

La théorie de régularité des surfaces minimales permet d'écrire l'équation d'Euler-Lagrange (1.3.2) de manière rigoureuse. La démonstration se généralise pour l'énergie $\mathcal{E}$ ci-dessus moyennant des résultats de régularité sur les quasi-minimiseurs du périmètre anisotrope. De tels résultats existent et sont établis dans le langage des courants.

Cependant, la régularité $C^{1,\alpha}$, $\alpha < 1/2$, fournie par cette théorie n'est pas suffisante. Il est possible de gagner un peu en utilisant la méthode des translations de Nirenberg. On peut alors poursuivre, en appliquant un résultat de régularité pour les EDPs elliptiques, pour obtenir *in fine* une régularité $W^{2,p}$ pour le bord réduit de E_t.

Dans le cas du problème de débruitage, le Théorème 1.3.1 stipule essentiellement que la minimisation de la variation totale ne produit pas de nouvel objet dans l'image. Cependant, cette assertion tombe en défaut si on s'intéresse au problème à poids

$$\min_{u \in BV(\Omega)} \int_\Omega w|Du| + \frac{1}{2}\|u - g\|_2^2 \tag{1.3.3}$$

où w est lipschitzienne et $\nabla w \in BV(\Omega, \mathbb{R}^N)$. Dans ce cas, il est possible d'observer la création de discontinuités absentes dans l'image g. En fait, on peut raffiner le résultat du théorème précédent :

Théorème 1.3.2. *Soit $w : \Omega \to \mathbb{R}$ un poids lipschitzien tel qu'il existe une constante $C_w > 0$ avec $C_w^{-1} \leq w \leq C_w$ et tel que $\nabla w \in BV(\Omega, \mathbb{R}^N)$. Soit $g \in BV(\Omega) \cap L^\infty(\Omega)$. Si on note $J_{\nabla w} := \bigcup_{i=1}^N J_{\partial_{x_i}} w$, alors le minimiseur u de (1.3.3) satisfait*

$$J_u \subset J_g \cup J_{\nabla w} \tag{1.3.4}$$

à un ensemble de mesure $\mathcal{H}^{N-1}$ négligeable près.

De plus, on dispose d'une borne sur la courbure moyenne de l'ensemble de saut

$$\kappa_{J_u} \in \left[C_w^{-1}(g^- - u^-), C_w(g^+ - u^+)\right] \ \mathcal{H}^{N-1}\text{-}p.p.$$

Enfin, si on suppose que w est de classe C^1, on obtient à la discontinuité

$$(u^+ - u^-) \leq (g^+ - g^-) \ sur\ J_u \ \ \mathcal{H}^{N-1}\text{-}p.p. \tag{1.3.5}$$

Ce théorème montre qu'il est possible de forcer la création de discontinuités en choisissant un poids w adéquat. Quant à (1.3.5), l'inégalité caractérise une baisse de contraste au niveau de la discontinuité et ce même pour une donnée qui oscille dans un voisinage de la discontinuité ! Ce résultat est utile dans [50] pour étendre les résultats de [48].

Dans le cas à poids, le résultat de régularité sous-jacent se démontre sans nécessairement faire appel à la théorie des courants. En effet, comme nous le verrons plus tard, un quasi-minimiseur du périmètre à poids est un quasi-minimiseur du périmètre classique. Ainsi, la théorie classique de [147] s'applique.

Création de zones homogènes

Dans cette partie, nous nous intéressons à la création de régions constantes, inhérente à la minimisation de la variation totale. Nous mettons en évidence la présence du phénomène de staircasing dans un cadre continu et en dimension $N \geq 2$. Comme nous l'avons expliqué précédemment, les seuls résultats de ce type sont énoncés soit dans un cadre discret, soit en dimension $N = 1$.

La contribution principale de cette section est énoncée dans le théorème suivant :

Théorème 1.3.3. (i) *Soient $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ et $\lambda > 0$. Alors le minimiseur u_λ de (1.2.4) est borné, atteint ses bornes et*

$$|\{u_\lambda = \min u_\lambda\}|, |\{u_\lambda = \max u_\lambda\}| > 0.$$

En particulier, $Du_\lambda = 0$ sur $\{u_\lambda = \min u_\lambda\} \cup \{u_\lambda = \max u_\lambda\}$ qui admet un représentant ouvert.

(ii) *Soient $g \in L^p(\Omega)$ pour un $p \in (N, +\infty]$, $\lambda > 0$ et x_0 un extremum local de u_λ. Alors, il existe un voisinage $\mathcal{N}(x_0)$ tel que*

$$Du_\lambda = 0 \text{ sur } \mathcal{N}(x_0).$$

Pour le premier point, nous proposons une démonstration simple qui repose uniquement sur la formule de la coaire. Quant au deuxième résultat, il utilise le théorème de densité pour les quasi-minimiseurs du périmètre.

Problème de débruitage pour une donnée radiale

Eu égard aux théorèmes précédemment établis, on peut désormais s'attacher à déterminer le comportement du minimiseur de l'énergie de Rudin, Osher et Fatemi en fonction du paramètre de régularisation λ.

Nous poussons donc notre étude un peu plus loin en démontrant que la solution du problème de ROF pour une donnée radiale g s'obtient en suivant le flot de la variation totale à partir de la donnée g. En d'autres termes, on a :

Théorème 1.3.4. *Soient $\Omega = B(0,R)$, $g \in L^2(\Omega)$ radiale et $u(t)$ la solution de*

$$-\partial_t u(t) \in \partial TV(u(t)) \text{ presque tout } t \in [0,T], \tag{1.3.6}$$

pour la condition initiale $u(0) = g$. Alors, $u(t)$ est l'unique minimiseur de

$$\min_{u \in BV(\Omega)} t \int_\Omega |Du| + \frac{1}{2}\|u-g\|_2^2. \tag{1.3.7}$$

En particulier, ce théorème impose une décroissance de l'ensemble de saut et une croissance de l'ensemble de staircase en fonction du paramètre de régularisation :

Théorème 1.3.5. *Soient $\Omega = B(0,R)$ et $g \in L^N(\Omega)$ radiale. Considérons deux réels $0 \leq \lambda < \mu$ et u_λ, u_μ deux minimiseurs de (1.2.4). Si on note S_λ et S_μ les ensembles de staircase respectifs, on a*

$$S_\lambda \subset S_\mu,$$
$$J_{u_\mu} \subset J_{u_\lambda} \subset J_g.$$

Une fois le Théorème 1.3.4 établi, l'emboîtement des sauts est une conséquence de [48] où Caselles, Chambolle et Novaga établissent ce résultat pour le flot.

Il serait intéressant de comprendre si le résultat d'inclusion pour les sauts reste vrai pour une donnée générale. En effet, la construction de solutions explicites par Allard [3] montre que le Théorème 1.3.4 n'est plus vrai lorsque la donnée n'est pas radiale. Le même exemple d'Allard montre qu'on ne peut rien espérer de tel pour l'ensemble de staircase.

1.3.2 Étude des propriétés d'une variante de la variation totale

Dans le Chapitre 3, nous proposons et étudions une alternative à la variation totale pour la restauration d'images. Une partie des résultats présentés a été publiée dans [100, 101].

Le terme de régularisation proposé se définit comme suit :

$$\mathcal{R}(u) = \inf\left\{\int_\Omega |Du+\psi| \ / \ \psi \in \mathcal{M}_b(\Omega,\mathbb{R}^N) \text{ et } \int_\Omega \nabla v \cdot \psi = 0 \ \forall v \in C^1(\bar{\Omega})\right\},$$

où $\mathcal{M}_b(\Omega,\mathbb{R}^N)$ est l'espace des mesures de Radon bornées à valeurs vectorielles. Remarquons immédiatement que $\mathcal{R}(u) \leq \int_\Omega |Du|$, donc cette fonctionnelle a bien un intérêt dans le contexte des fonctions de classe BV.

Grâce à un argument de dualité convexe, il est possible de prouver que

$$\mathcal{R}(u) = \sup \left\{ \int_\Omega \nabla w \cdot Du, \; w \in C^1(\bar{\Omega}) \text{ et } \|\nabla w\|_\infty \leq 1 \right\}.$$

Cette réécriture de la définition montre que le calcul de $\mathcal{R}(u)$ fait appel aux dérivées secondes de u et permet de classer $\mathcal{R}$ dans la catégorie des régularisations d'ordre supérieur (cf. la section 1.2.8).

La formulation duale est une étape importante pour démontrer le théorème suivant :

Théorème 1.3.6. *Soient $\Omega \subset \mathbb{R}^N$ ouvert et $u = \chi_E$ la fonction caractéristique d'un ensemble de périmètre fini E dans Ω, ou plus généralement $u \in BV(\Omega)$ avec une dérivée Du concentrée sur l'ensemble de saut J_u. Alors,*

$$\mathcal{R}(u) = \int_\Omega |Du|.$$

La preuve repose sur le fait que l'ensemble de discontinuité d'une fonction de classe BV est rectifiable. Rappeleons qu'un ensemble est rectifiable lorsqu'il admet des hyperplans tangents en un sens faible.

Le Théorème 1.3.6 légitime l'utilisation de $\mathcal{R}$ dans le contexte du traitement des images puisque, en somme, cette fonctionnelle coïncide avec la variation totale dans le cas des images de type "cartoon".

Cette proximité avec la variation totale se traduit également par l'inégalité de type Poincaré suivante :

Proposition 1.3.7. *Soient $\Omega \subset \mathbb{R}^N$ ouvert, $u \in BV(\Omega)$ et $(\rho_\varepsilon)_{\varepsilon>0}$ un noyau régularisant radial. Alors, pour $K \subset\subset \Omega$ et $\varepsilon > 0$ petit, on a*

$$\|u * \rho_\varepsilon - u\|_{L^1(K)} \leq C\varepsilon\mathcal{R}(u).$$

Cette proposition est une première étape pour démontrer une inégalité de Poincaré dans ce contexte.

Dans la dernière partie de ce chapitre, nous revisitons le problème de Rudin, Osher et Fatemi en adoptant la régularisation proposée. Après avoir démontré l'existence et l'unicité d'une solution, nous calculons la solution explicite d'une donnée qui est la caractéristique d'un disque et présentons des tests comparatifs avec la variation totale.

1.3.3 Régularisation non-locale pour la complétion de spectre

Le Chapitre 4 (dont les résultats proviennent de [56]) est consacré à l'étude de divers problèmes inverses où la donnée vient sous la forme de coefficients de Fourier corrompus . Ce type de problèmes trouve des applications dans des domaines variés.
Donnons quelques exemples :

(i) Le problème inverse de diffusion d'ondes acoustiques (ou *scattering inverse problem*) sert à la détection d'objets enfouis dans un matériau. Il s'agit d'un problème récurrent en géophysique et en imagerie médicale. Dans le cadre de l'approximation de Born, nous devons déterminer l'objet représenté par l'ensemble D à partir de la donnée du champ de diffusion lointain

$$u_\infty(x,d) = \int_{\mathbb{R}^N} \chi_D(y) e^{-i(x-d)\cdot y} dy,$$

où x et d appartiennent à la sphère unité. Ceci se ramène à un problème général d'échantillonage du spectre. En effet, dans ce cas, $u_\infty(x,d)$ peut être interprété comme un coefficient de Fourier.

(ii) Le problème de tomographie en imagerie médicale : dans ce cas les coefficients de Fourier sont connus le long de droites qui sont soit parallèles, soit qui se croisent à l'origine. L'IRM (Imagerie par Résonance Magnétique) utilise cette technique.

(iii) En imagerie spatiale et satellitaire, la "synthèse d'ouverture" ("*aperture synthesis*" en anglais) est un procédé qui permet de regrouper les images issues d'un ensemble de télescopes pour produire une image qui a la même résolution que celle produite par un télescope ayant la taille de l'ensemble. L'image fournie par chaque télescope est une composante de la transformée de Fourier de l'image finale. Le Very Large Array au Nouveau Mexique regroupe un ensemble de 27 radiotélescopes et permet de simuler un radiotélescope d'un diamètre de 36km.

(iv) Le problème de zoom en traitement des images (utile par exemple pour le passage de la SD (*Standard Definition*) à la HD (*High Definition*)) se ramène à la construction de hautes fréquences cohérentes avec une donnée basse fréquence.

L'objet de notre étude a été de trouver un critère adéquat pour régulariser ce type de problèmes. L'objectif étant d'élaborer une énergie non-locale du type (1.2.9) pour exploiter autant que possible les redondances de la donnée.

Dans ce type d'approches, il est important de comprendre à partir de l'image dégradée, les régions de l'image qui étaient semblables dans l'image d'origine. Dans les méthode de type NLMeans (comprenant donc l'énergie (1.2.9)) le critère adopté est la distance ℓ^2. Ainsi deux voisinages $p_{g_0}(x_1)$, $p_{g_0}(x_2)$, centrés en les pixels x_1 et x_2, étaient proches dans l'image d'origine si la quantité

$$\|p_g(x_1) - p_g(x_2)\|_2,$$

calculée sur l'image dégradée g, est petite.

L'idée maîtresse de notre travail consiste à proposer un critère qui quantifie la proximité de deux régions de l'image dégradée et qui diffère de celui utilisé pour les NLMeans. Il s'agit de construire une famille de fonctions tests $(\phi_\alpha)_\alpha$ adaptée au processus de corruption telle que

$$g * \phi_\alpha = g_0 * \phi_\alpha, \ \forall \alpha.$$

Si les atomes ainsi construits sont suffisamment nombreux et ont un support réduit, il est possible de supputer que deux voisinages centrés en x_1 et x_2 étaient proches dans l'image d'origine si la quantité

$$\sum_\alpha |g * \phi_\alpha(x_1) - g * \phi_\alpha(x_2)|^2,$$

calculée à partir de l'image dégradée, est petite. Nous expliquerons en détail la construction de ces atomes et comparerons notre approche avec la méthode classique.

1.3.4 Optimisation convexe : aspects algorithmiques

Dans le dernier chapitre de cette thèse, nous nous penchons sur les aspects algorithmiques inhérents à la minimisation des énergies convexes susmentionnées. L'élaboration d'algorithmes rapides est un point crucial si l'on veut espérer procéder, par exemple, à des traitements en temps réel dans des dispositifs embarqués. Enfin le développement du calcul parallèle sur GPU impose de nouveaux défis dans ce domaine. Nous rappelons à l'occasion de ce chapitre quelques avancées récentes et consacrons une section importante aux approches dites “Primal-Dual”.

Récemment, cette nouvelle famille d'algorithmes a été envisagée en traitement d'images pour la recherche d'un point-selle $(\hat{x}, \hat{y})$ d'un problème général du type

$$\min_{x \in X} \max_{y \in Y} \langle Ax, y \rangle + G(x) - F^*(y)$$

où X et Y sont deux espaces vectoriels de dimension finie, $A : X \to Y$ est une matrice, F^* désigne la transformée de Legendre de la fonction convexe sci F et G en plus d'être sci est uniformément convexe de paramètre γ.

Quitte à introduire une variable dite “duale”, cette structure générique s'adapte à de nombreux problèmes de minimisation en traitement des images. On peut notamment penser à la minimisation de la variation totale ou bien simplement de la norme L^1, primordiale pour avoir une représentation parcimonieuse. L'approche naïve qui consiste à effectuer une descente de gradient est rendue difficile puisque les énergies considérées ne sont pas différentiables. L'introduction de la dualité à permis dès la fin des années 90 de contourner ce problème [62, 54].

L'algorithme de type Primal-Dual proposé dans [159, 59, 82] s'inspire d'un travail ancien de Arrow et Hurwicz et consiste à faire une descente de gradient en la variable primale x et une montée de gradient en la variable duale y :

Algorithme 1.1 Schéma d'Arrow-Hurwicz

- **Initialisation :** Choisir $x^0 \in X$, $y^0 \in Y$, $(\sigma_n)_n, (\tau_n)_n$.
- **Itérations :** Pour $n \geq 1$, mettre à jour comme suit :

$$x^{n+1} = (I + \tau_n \partial G)^{-1}(x^n - \tau_n A^* y^n),$$
$$y^{n+1} = (I + \sigma_n \partial F^*)^{-1}(y^n + \sigma_n A x^{n+1}).$$

Rappelons que $(I + \partial H)^{-1}$ désigne l'opérateur proximal introduit par Moreau.

Dans le cas où les pas σ_n et τ_n sont maintenus constants, nous démontrons que l'algorithme précédent permet de construire une suite minimisante convergente et que l'on a une estimation du taux de convergence :

Théorème 1.3.8. *Soient $\tau, \sigma > 0$ tels que $\sigma \leq \frac{\gamma}{\|A\|^2}$. Alors la suite $(x^n)_{n\in\mathbb{N}}$ converge vers $\hat{x}$ et l'erreur $\|\hat{x} - x^n\|^2$ est sommable.*

Chambolle et Pock [59] s'intéressent au cas à pas variables et montrent qu'une modification simple de cet algorithme offre un résultat de convergence en $O\left(\frac{1}{n^2}\right)$. Cependant, une observation empirique montre que le choix du paramètre de convexité γ est crucial dans leur algorithme et donne lieu à des résultats très contrastés. Cette remarque nous amène à proposer une variante de leur preuve qui améliore le taux de convergence connu :

Théorème 1.3.9. *Étant donnés $\tau_0, \sigma_0 > 0$ tels que $\sigma_0 \tau_0 \|A\|^2 \leq 1$ alors la suite $(x^n)_{n\in\mathbb{N}}$ converge vers $\hat{x}$ et*

$$\sum_n n\|\hat{x} - x^n\|^2 < +\infty.$$

Ceci laisse espérer une convergence en $o\left(\frac{1}{n^2}\right)$ observée dans la pratique et qui dépasse donc la complexité théorique établie dans [21, 131, 132, 59] mais également la limite théorique en $O\left(\frac{1}{n^2}\right)$ de [130] pour les algorithmes de premier ordre et qui est valable pour des problèmes convexes généraux.

Dans la dernière partie de ce chapitre, nous proposons un comparatif de la performance des algorithmes couramment utilisés pour le problème de débruitage par variation totale.

CHAPTER 2

Fine Properties of the Total Variation Minimization Problem

Contents

2.1 Introduction

Functions of bounded variation equipped with the total variation semi-norm were introduced for image reconstruction in 1992. Since then, they have had many successful applications for inverse problems in imaging. Indeed, the penalization of the total variation has the ability to smooth out the image by creating large regular zones and to keep the edges of the most important objects in the image. In this chapter we aim to study these two key properties in the continuous setting and for general energies.

We assume that a corrupted image $g : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ went through a degradation

$$g = g_0 + n$$

where g_0 is the original clean image, n is a Gaussian white noise of standard deviation σ. Rudin, Osher and Fatemi (ROF) proposed in 1992 to minimize the total variation

$$u \mapsto TV(u) = \int_\Omega |Du|$$

amongst functions of bounded variation under the constraint $\|u - g\|_2^2 \leq \sigma^2 |\Omega|^2$ to solve the inverse problem and thus get a restored image u. It was proven in [57] that one can solve in an equivalent way the unconstrained problem

$$\min_{u \in BV(\Omega)} \lambda \int_\Omega |Du| + \frac{1}{2}\|u - g\|_2^2$$

for an adequate Lagrange multiplier λ. In the literature the minimization of ROF's energy is referred to as the *denoising problem*.

Another possibility is to consider the *total variation flow* for restoring g. A denoised image is given by $u(t)$ that solves

$$\begin{cases} -\partial_t u(t) \in \partial TV(u(t)) \text{ a.e. } t \in [0, T], \\ u(0) = g. \end{cases}$$

which has a unique solution according to [13]. As we shall see it is not true in general that these approaches coincide.

It has been long observed that using the total variation has the advantage of recovering the discontinuities quite well. We will devote a part of our study to the behavior of the minimizer of the denoising problem at these discontinuities. Recently Chambolle, Caselles and Novaga proved in [48] that the discontinuities of the denoised image are contained in those of the datum g. In other words minimizing ROF's energy does not create new discontinuities. The idea of their proof is to use the coarea formula to look at the level sets of the minimizer locally and to detect the creation of jumps when two of these level sets touch. The argument was further refined by the same authors in [49] to prove local Hölder continuity of the minimizer of the denoising problem when the datum is itself Hölder continuous. All these results are extended to the case of the total variation flow in both papers. See also [50] where the initial datum g is not assumed to be bounded.

In the first part of this work, our aim is to generalize their results to a problem of the form

$$\min_{u \in BV(\Omega)} \int_\Omega \Phi(x, Du) + \int_\Omega \Psi(x, u(x))dx.$$

Here Φ is a smooth elliptic anisotropy, Ψ is essentially strictly convex, coercive in the second variable and integrable in the first one. To adapt the argument of [48] we need to recall some basic results on the regularity of solutions of elliptic PDEs and also some standard facts on the regularity of minimal surfaces [144].

Moreover, we refine the obtained result in the weighted case

$$\min_{u \in BV(\Omega)} \int_\Omega w(x)|Du| + \frac{1}{2}\|u - g\|_2^2$$

and prove that whenever w is merely Lipschitz, one can observe the creation of discontinuities, that is to say, the minimizer has discontinuities that are not contained in those of the datum g. We also prove that the jump (think of the contrast for images) is decreased at the discontinuity. This is quite counterintuitive is one consider a datum that is highly oscillating in the nerighborhood of the discontinuity. Our result is a key step in [50], allowing that the authors to extend the results of [48]. In the weighted case, we also prove directly the regularity of the level lines. This way one can avoid using the difficult result of [144] that makes a wide usage of the theory of currents.

In the second part of this chapter, we shall focus on another very important property of the total variation: it smoothes the highly oscillating regions by creating large constant zones which is known in the literature as the *staircasing effect*. This phenomenon is sometimes not desirable in imaging applications since it yields blocky and non-natural structures. It was already studied in [140] for the one-dimensional case. Actually, the author proves that whenever the data $g \notin BV(a, b)$, the minimizer u'_λ vanishes almost everywhere. In [133], Nikolova proves that the staircasing effect is related to the non-differentiability of the total variation term. More precisely, large homogeneous zones are recovered from noisy data and remain unchanged for small perturbations. In other words, the creation of such zones is quite probable. On the contrary, absence of staircasing with the differentiable approximation $TV_\varepsilon = \sqrt{|Du|^2 + \varepsilon}$ is also proven. The latter approaches were carried out only for finite dimensional approximations of the total variation. In [109], Louchet proposed an alternative to the minimization of the total variation by considering the TV-LSE filter. She proved in the discrete setting that TV-LSE avoids the staircasing effect (in the sense that a region made of 2 pixels or more where the restored image is constant almost never occurs).

As far as we know, there is no result on the subject in the higher dimensional and continuous setting for the classical total variation functional. We shall show that staircasing always occurs (even without addition of noise) at global extrema of the datum and at all extrema of the minimizer.

In the last section, we investigate further these qualitative properties. An interesting question is to understand how the staircase zones and the discontinuities evolve with the regularization parameter λ. The idea is to use the results that are already established for the flow. Unfortunately, in higher dimension the connection between the flow and ROF's energy fails (we give a counterexample). Though, we are going to prove that this connection actually holds for radial functions. This way, one can prove that the discontinuities form a decreasing sequence, whereas the staircase zones increase with the regularization parameter (which is not true in general).

Let us remark that all the results are established in dimension $N \geq 2$ since the situation is quite well understood in the one-dimensional case and was widely studied in the literature (see the recent paper [33] for instance). Indeed, in dimension one, ROF's denoising problem reads as follows

$$\min_{u \in BV(\mathbb{R})} \int \lambda |u'(x)| + \frac{1}{2}(u-g)^2(x)dx \tag{2.1.1}$$

for $g \in L^2(\mathbb{R})$ and some positive real λ. Let us denote u_λ the minimizer of this problem.

Writing down the Euler-Lagrange (see equation 2.2.2) one immediately sees that either u_λ is constant or $z_\lambda = \operatorname{sgn}(u_\lambda')$ and as a consequence $u_\lambda = g$. This is an almost explicit formulation of the solution that tells us that

- the discontinuities of u_λ are contained in those of g,
- flat zones are created at maxima and minima of g.

This can be seen in the following simulation:

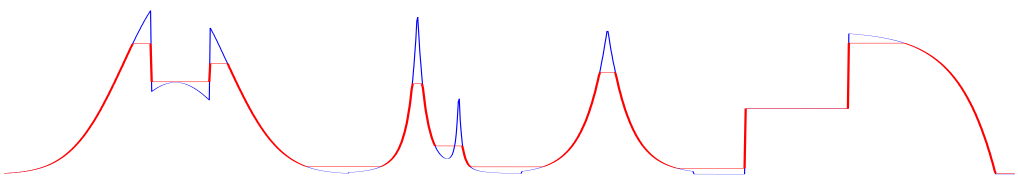


Figure 2.1: Minimizer u_λ (in red) of a 1D data g (in blue).

The other important result in dimension one is the link with the flow of the total variation. It is known that problem (2.1.1) with $\lambda = t$ is minimized by $u(t)$, the unique solution of the flow. In the recent article [43], the authors used this relation to prove that for a signal that went through an addition of noise (that is the trajectory of a Wiener process strictly speaking), staircasing occurs almost everywhere. This observation seems to be more general as can be seen in the following test:

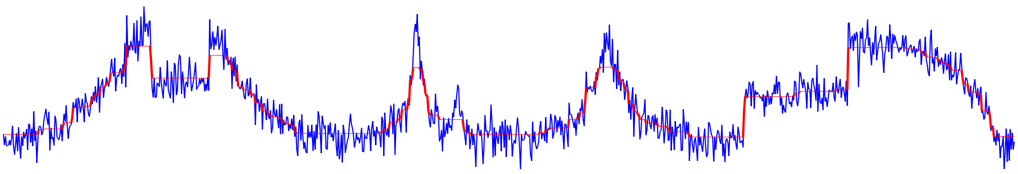


Figure 2.2: Minimizer u_λ (in red) of a noisy data g (in blue) is constant almost everywhere.

2.2 Mathematical preliminary

Henceforth Ω will denote an open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary. The material of this section can be found in the classical textbooks [8, 92, 160] but also in the recent survey [55].

2.2.1 Functions of bounded variation

Definition 2.2.1. A function $u \in L^1(\Omega)$ is of *bounded variation* in Ω (denoted $u \in BV(\Omega)$) if its distributional derivative Du is a vector-valued Radon measure that has finite total variation *i.e.* $|Du|(\Omega) < \infty$. By the Riesz representation theorem, this is equivalent to say that

$$|Du|(\Omega) = \sup \left\{ \int_\Omega u \operatorname{div} \varphi \ / \ \varphi \in C_c^\infty(\Omega, \mathbb{R}^N), \forall x \in \Omega \ |\varphi(x)| \leq 1 \right\} < \infty.$$

In the sequel, the quantity $|Du|(\Omega)$ also denoted $\int_\Omega |Du|$ or simply $TV(u)$ will be called the *total variation* of u. It is readily checked that $\|\cdot\|_1 + TV$ defines a norm on $BV(\Omega)$ that makes it a Banach space.

A first result that is a straightforward consequence of the dual definition we just gave is a key step to apply the direct method in the calculus of variations:

Proposition 2.2.2 (Sequential lower semicontinuity)**.** *Let $(u_n)_{n\in\mathbb{N}}$ be any sequence in $BV(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ then*

$$\int_\Omega |Du| \leq \liminf_{n\to\infty} \int_\Omega |Du_n|.$$

One also has

Proposition 2.2.3 (Approximation by smooth functions). *If $u \in BV(\Omega)$ then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $C^\infty(\Omega)$ such that*

$$u_n \to u \text{ in } L^1(\Omega)$$
$$\int_\Omega |\nabla u_n| \to \int_\Omega |Du|.$$

For the direct method to apply it is usually of importance to have a compactness result:

Theorem 2.2.4 (Rellich's compactness in BV). *Given a bounded $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary and any sequence $(u_n)_{n\in\mathbb{N}}$ such that $\left(\|u_n\|_{L^1(\Omega)} + \int_\Omega |Du_n|\right)$ is bounded, there exists a subsequence $(u_{n(k)})_{k\in\mathbb{N}}$ that converges in L^1 to some $u \in BV(\Omega)$ as $k \to \infty$.*

Definition 2.2.5. Let $E \subset \mathbb{R}^N$ be a Borelian set. It is called a *set of finite perimeter* or also *Cacciopoli set* if $u = \chi_E$ is a function of bounded variation. We will call *perimeter* of E in Ω, and denote $P(E,\Omega)$ or simply $P(E)$, its total variation.

The following key result provides a connection between the total variation of a function and the perimeter of its level sets.

Theorem 2.2.6 (Coarea formula). *If $u \in BV(\Omega)$, the set $E_t = \{u > t\}$ has finite perimeter for a.e. $t \in \mathbb{R}$ and*

$$|Du|(B) = \int_{-\infty}^{\infty} |D\chi_{\{u>t\}}|(B)dt$$

for any Borel set $B \subset \Omega$.

Functions of bounded variation have some nice structural properties that we are going to recall here:

Definition 2.2.7. We say that $u \in L^1_{loc}(\Omega)$ has an *approximate limit* at $x \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y) - z|dy = 0.$$

The set of points where this does not hold is called the *approximate discontinuity set* and denoted S_u.

We say that $x \in \Omega$ is an *approximate jump point* of u if there exist $u^+(x) \neq u^-(x) \in \mathbb{R}$, $\nu(x) \in \mathbb{R}^N$ a unitary vector such that

$$\lim_{r\to 0} \frac{1}{|B_r^\pm(x,\nu(x))|} \int_{B_r^\pm(x,\nu(x))} |u(y) - u^\pm(x)| = 0$$

where $B_r^{\pm}(x, \nu(x)) = \{y \in B(x,r); \pm|\langle \nu(x), y-x\rangle| > 0\}$. We shall denote by J_u the set of *jump points*.

If $u = \chi_E$ is the characteristic function of a set E of finite perimeter in Ω, J_u is then denoted $\partial^* E$ and called the *reduced boundary* of E.

Then we have the following structure theorem that will be further detailed in Chapter 3 (see Section 3.5).

Theorem 2.2.8. *If $u \in BV(\Omega)$ then*

$$\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$$

and one has the following decomposition

$$Du = \nabla u dx + (u^+ - u^-)\nu \mathcal{H}^{N-1}_{|J_u} + D^c u$$

for some measure $D^c u$ referred to as the Cantor part of Du.

Finally let us recall the following

Theorem 2.2.9 (Sobolev inequalities)**.** *Let $u \in BV(\Omega)$ and let us denote $\langle u\rangle = \frac{1}{|\Omega|}\int_\Omega u$. For a bounded Lipschitz domain Ω, the following Poincaré inequality holds*

$$\|u - \langle u\rangle\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C(N,\Omega)\int_\Omega |Du|.$$

If $\Omega = \mathbb{R}^N$, one has the Sobolev inequality

$$\|u\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq C(N)\int_\Omega |Du|.$$

In particular, if $u = \chi_E$, one gets the isoperimetric inequality

$$|E|^{\frac{N-1}{N}} \leq C(N)P(E,\Omega).$$

2.2.2 BV functions in image processing

The classical model of a functional where total variation plays a key role is the so-called Rudin-Osher-Fatemi energy:

$$\mathcal{E}_\lambda(u) = \lambda\int_\Omega |Du| + \frac{1}{2}\|u-g\|_2^2 \tag{ROF}$$

In the sequel we shall be interested in minimizing this energy in $BV(\Omega)$ for some positive real λ. By Proposition 2.2.2, there is a unique minimizer in $BV(\Omega)$, denoted u_λ in the sequel. The parameter λ really plays the role of a tuning parameter as one can see it in the following

Proposition 2.2.10. *Let $g \in L^2(\Omega)$, λ some positive real and u_λ be the corresponding minimizer of (ROF) then whenever $\lambda \to 0$*

$$u_\lambda \to g \text{ in } L^2(\Omega).$$

In other words, the less we regularize the closer the minimizer gets to the data in the L^2 sense. This is quite what we expect. The proof is really simple in case $g \in BV(\Omega)$: since g is itself a candidate for the minimization

$$\lambda \int_\Omega |Du_\lambda| + \frac{1}{2}\|u_\lambda - g\|_2^2 \leq \lambda \int_\Omega |Dg| \tag{2.2.1}$$

which yields the result with $\lambda \to 0$. In case $g \in L^2(\Omega)$, this proposition is actually a basic property of the proximal mapping (see [41, Proposition 2.6]).

The energy $\mathcal{E}_\lambda$ is not smooth though convex so it is still possible to get the Euler-Lagrange equation as follows:

Proposition 2.2.11. *Function u_λ minimizes $\mathcal{E}_\lambda$ in $BV(\Omega)$ if and only if there exists $z_\lambda \in L^\infty(\Omega, \mathbb{R}^N)$ such that*

$$\begin{cases} -\lambda \operatorname{div} z_\lambda + u_\lambda = g & \text{in } \Omega, \\ |z_\lambda| \leq 1 & \text{in } \Omega, \\ z_\lambda \cdot Du_\lambda = |Du_\lambda|, & \\ z_\lambda \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.2}$$

with ν denoting the inner normal to Ω.

In the sequel, we shall only consider Neumann boundary conditions but we could of course take into account Dirichlet or more complicated conditions.

Remark 2.2.12. $z_\lambda \cdot Du_\lambda$ is the pairing of a bounded function with a bounded measure and should be understood in the sense of Anzelotti [14].

By the coarea formula, the superlevel sets $\{u_\lambda > t\}$ are sets of finite perimeter for almost every t that satisfy the following minimal surface problem:

Theorem 2.2.13. *Let u_λ be the minimizer of (ROF). Then for any $t \in \mathbb{R}$, $\{u_\lambda > t\}$ (resp. $\{u_\lambda \geq t\}$) is the minimal (resp. maximal) solution of the minimal surface problem*

$$\min_E \lambda P(E, \Omega) + \int_E (t - g(x))\, dx \tag{2.2.3}$$

over all sets of finite perimeter in Ω. Moreover $\{u_\lambda > t\}$ being defined up to negligible sets, there exists an open representative.

The results of this section form the foundations for the study of similar properties for more general energies. This will be the object of the next part.

2.3 Anisotropic total variation with a generic data fidelity

2.3.1 The anisotropic total variation: the general case of a Finsler metric

In calculus of variations, one is often interested in minimizing an integral functional of the form

$$\mathcal{E}(u) = \int_\Omega F(x, u, \nabla u)$$

among all $C^1(\Omega)$ or $W^{1,1}(\Omega)$ functions u and under some additional constraints. Though, the minimization problem is not well-posed and it is natural to seek for an extension of this functional in the completion of $W^{1,1}(\Omega)$ which is the space $BV(\Omega)$. Moreover, to apply the classical direct method of the calculus of variations we need the extension functional to be lower semicontinuous with respect to the L^1 convergence. The natural choice for the extension is therefore the so-called *relaxed functional* $\bar{\mathcal{E}}(u)$ which corresponds to the lower semicontinuous envelope.

In the sequel, we shall be interested in minimizing energies of the form

$$\mathcal{E}(u) = \int_\Omega \Phi(x, \nabla u(x))dx + \int_\Omega \Psi(x, u(x))dx$$

for some Φ and Ψ that we will specify later. For the moment, we are going to focus on the first term namely

$$J_\Phi(u) = \int_\Omega \Phi(x, \nabla u(x))dx$$

and will recall its lower semicontinuous envelope $\bar{J}_\Phi$.

From now on, the integrand $\Phi(x,p) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ will be called *a Finsler integrand* if

(H1) $\Phi(x, \cdot)$ is convex for any $x \in \Omega$,

(H2) $\Phi(x, \cdot)$ is of linear growth uniformly $x \in \Omega$ *i.e.*

$$C_\Phi^{-1}|p| \leq \Phi(x,p) \leq C_\Phi|p|, \ \forall x \in \Omega, \ p \in \mathbb{R}^N$$

for some positive real constant C_Φ,

(H3) Φ is positively 1-homogeneous in the variable p *i.e.*

$$\Phi(\cdot, \lambda p) = \lambda \Phi(\cdot, p), \ \forall \lambda > 0, \ p \in \mathbb{R}^N,$$

(H4) Φ is continuous.

The integrand Φ is a *reversible Finsler integrand* if it satisfies in addition

(H5) $\Phi(x, -p) = \Phi(x, p), \ \forall x \in \Omega, \ p \in \mathbb{R}^N.$

If one assumes that

(H6) $\Phi(\cdot, p)$ and $D_p\Phi(\cdot, p)$ are Lipschitz continuous on Ω uniformly $p \in \mathbb{S}^{N-1}$ *i.e.*

$$\sup_{p \in \mathbb{S}^{N-1}} |\Phi(x, p) - \Phi(\tilde{x}, p)| \leq C|x - \tilde{x}| \ \forall x, \tilde{x} \in \Omega,$$
$$\sup_{p \in \mathbb{S}^{N-1}} |D_p\Phi(x, p) - D_p\Phi(\tilde{x}, p)| \leq C|x - \tilde{x}| \ \forall x, \tilde{x} \in \Omega,$$

(H7) $\Phi(x, \cdot)$ has locally β-Hölder second order partial derivatives (with $\beta \in (0, 1]$) on $\mathbb{R}^N \setminus \{0\}$ uniformly $x \in \Omega$ and

$$|D_p^2\Phi(x, p)| \leq C \ \forall x \in \Omega, \ p \in \mathbb{S}^{N-1},$$

(H8) Φ is *elliptic* in the sense that

$$\langle D_p^2\Phi(x, p)\xi, \xi\rangle \geq \frac{\left|\xi - \left(\xi \cdot \frac{p}{|p|}\right)\frac{p}{|p|}\right|^2}{|p|}, \ \forall x \in \Omega, \ \xi \in \mathbb{R}^N, \ p \in \mathbb{R}^N \setminus \{0\},$$

or equivalently (see [144])

$$\langle \nabla_p\Phi(x, p) - \nabla_p\Phi(x, \tilde{p}), p - \tilde{p}\rangle \geq |p - \tilde{p}|^2, \ \forall x \in \Omega, \ p, \ \tilde{p} \in \mathbb{S}^{N-1},$$

we shall say that Φ is *a strongly convex Finsler integrand.*

Remarks 2.3.1.(i) Assumptions $(H1) - (H3)$ imply that $\Phi(x, \cdot)$ is Lipschitz continuous uniformly $x \in \Omega$ *i.e.*

$$\sup_{x \in \Omega} |\Phi(x, p) - \Phi(x, \tilde{p})| \leq C|p - \tilde{p}|.$$

(ii) For later reference we also note that

$$p \cdot \nabla_p\Phi(\cdot, p) = \Phi(\cdot, p)$$

by assumption $(H3)$. If $\beta = 1$, then by $(H7)$ it follows that $\nabla_p \Phi$ is Lipschitz continuous on $\mathbb{R}^N \times \mathbb{S}^{N-1}$.

(iii) Any Riemannian metric or more generally Finsler metric gives rise to a Finsler integrand. We refer to [6, 22] for further details.

(iv) We get the total variation by simply setting $\Phi(x,p) = |p|$ which is also called in the literature the *area integrand.*

As we just said, to be of some interest the extension of functional J_Φ has to be lower semicontinuous on $BV(\Omega)$. This is ensured by the following representation result:

Proposition 2.3.2. *Let* $\Phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *be a Finsler integrand. For any* $u \in BV(\Omega)$ *we have*

$$\bar{J}_\Phi(u) = \int_\Omega \Phi(x, \nabla u) + \int_\Omega \Phi\left(x, \frac{D^s u}{|D^s u|}\right) |D^s u|$$

where $\frac{D^s u}{|D^s u|}$ *is the Radon-Nikodym derivative of* $D^s u$ *with respect to* $|D^s u|$.

One of the first versions of this theorem was proven by Demengel and Temam in [76] in case $J_\Phi(\mu) = \int \Phi(\mu)$. The result was progressively refined in articles such as [35] and [18] which contains the general case in which we are interested. We would like to note that the hypotheses we made can be weakened a little. It is indeed possible to assume that Φ is only a quasi-convex function in p, not necessarily continuous and we need a little less than the 1-homogeneity but for the sake of clarity we refer to [8] for the exact statement of the general theorem.

In [6, Theorem 5.1], it is proven that the latter definition has a dual counterpart:

Proposition 2.3.3. *Let* Φ *be a Finsler integrand and* $u \in BV(\Omega)$ *then*

$$\bar{J}_\Phi(u) = \sup\left\{ \int_\Omega u \operatorname{div}\varphi \;/\; \varphi \in C_c^1(\Omega, \mathbb{R}^N),\ \forall x \in \Omega\ \Phi^0(x, \varphi(x)) \leq 1 \right\}$$

where Φ^0 *denotes the polar of* Φ *defined by*

$$\Phi^0(x, \varphi(x)) = \max\{p \cdot \varphi(x) \;/\; p \in \mathbb{R}^N, \Phi(x,p) \leq 1\}.$$

Henceforth we will not make any distinction between J_Φ and its L^1-lower semicontinuous envelope $\bar{J}_\Phi$.

Definition 2.3.4. Let Φ be a Finsler integrand. If $u \in BV(\Omega)$ then the quantity $J_\Phi(u,\Omega)$ (or simply $J_\Phi(u)$ if $\Omega = \mathbb{R}^N$) is the *anisotropic total variation* of u in Ω. If E is a set of finite perimeter then the *anisotropic perimeter* of E in Ω, denoted $P_\Phi(E,\Omega)$ (or $P_\Phi(E)$ if $\Omega = \mathbb{R}^N$) is the anisotropic total variation of E namely

$$P_\Phi(E,\Omega) = \int_{\partial^* E \cap \Omega} \Phi(x, \nu_E) d\mathcal{H}^{N-1}(x).$$

Remark 2.3.5. If Φ is a reversible Finsler integrand then for any set E of finite perimeter in Ω one has

$$P_\Phi(E,\Omega) = P_\Phi(\Omega \setminus E, \Omega).$$

Soon, we will need two generalizations of the coarea formula for the anisotropic total variation. Let us state them here:

Proposition 2.3.6. *Let $u \in BV(\Omega)$ and $w : \Omega \to \mathbb{R}$ be a non-negative Borelian weight. Then one has*

$$\int_\Omega w|Du| = \int_{-\infty}^{\infty} \left(\int_\Omega w |D\chi_{\{u>t\}}| \right) dt = \int_{-\infty}^{\infty} P_w(\{u > t\}, \Omega) dt.$$

Proof. By [84, Theorem 7], there exists a sequence of Borelian sets $(A_k)_{k\in\mathbb{N}}$ such that

$$w = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}.$$

Therefore by Fubini and then by Theorem 2.2.6,

$$\begin{aligned}
\int_\Omega w|Du| &= \sum_{k=1}^{\infty} \frac{1}{k} \int_{A_k} |Du| = \sum_{k=1}^{\infty} \frac{1}{k} \int_{-\infty}^{\infty} |D\chi_{\{u>t\}}|(A_k) dt \\
&= \int_{-\infty}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k} \int_{A_k} |D\chi_{\{u>t\}}| \right) dt = \int_{-\infty}^{\infty} \left(\int_\Omega w |D\chi_{\{u>t\}}| \right) dt.
\end{aligned}$$

□

The following proposition is stated in [6, Remark 4.4] without any proof:

Proposition 2.3.7. *Let $u \in BV(\Omega)$ and $\Phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Finsler integrand. Then then one has*

$$\int_\Omega \Phi(x, Du) = \int_{-\infty}^{\infty} P_\Phi(\{u > t\}, \Omega) dt.$$

Proof. Applying Proposition 2.2.3, one can pick an approximating sequence $(u_n)_{n\in\mathbb{N}}$ of C^∞ functions such that $u_n \to u$ in $L^1(\Omega)$ and $\int_\Omega |\nabla u_n| \to \int_\Omega |Du|$.

If we set

$$w(x) = \Phi\left(x, \frac{\nabla u_n(x)}{|\nabla u_n(x)|}\right)$$

whenever $\nabla u_n(x) \neq 0$ then by Proposition 2.3.6 the result holds for u_n, namely

$$\begin{aligned}\int_\Omega \Phi\left(x, \frac{\nabla u_n(x)}{|\nabla u_n(x)|}\right)|\nabla u_n(x)|dx &= \int_{-\infty}^{\infty}\left(\int_\Omega \Phi\left(\cdot\,, \frac{\nabla u_n}{|\nabla u_n|}\right)|D\chi_{\{u_n>t\}}|\right)dt \\ &= \int_{-\infty}^{\infty} P_\Phi(\{u_n > t\}, \Omega)dt.\end{aligned}$$

Finally by Reshetnyak Theorem 2.39 in [8], we can send $n \to +\infty$ and we get

$$\begin{aligned}\int_\Omega \Phi\left(x, \frac{Du}{|Du|}\right)d|Du| &\geq \liminf_{n\to\infty}\int_{-\infty}^{\infty} P_\Phi(\{u_n > t\}, \Omega)dt \\ &\geq \int_{-\infty}^{\infty} P_\Phi(\{u > t\}, \Omega)dt\end{aligned}$$

where in the second line we used the lower semicontinuity of P_Φ in conjunction with Fatou's lemma.

To prove the converse inequality, let us pick a candidate $\varphi \in C_c^1(\Omega, \mathbb{R}^N)$ such that for any $x \in \Omega$, $\Phi^0(x, \varphi(x)) \leq 1$. Then, by the layer cake formula and by application of Fubini and Proposition 2.3.3,

$$\begin{aligned}\int_\Omega u \operatorname{div}\varphi &= \int_{-\infty}^{\infty}\int_\Omega \chi_{\{u>t\}}(x)\operatorname{div}\varphi(x)\ dxdt \\ &\leq \int_{-\infty}^{\infty} P_\Phi\left(\{u > t\}, \Omega\right)dt,\end{aligned}$$

which proves the result taking the supremum of the left hand side over all admissible φ.

□

2.3.2 The minimization problem for functions

In the sequel, we are going to consider the following energy

$$\mathcal{E}(u) = \int_\Omega \Phi(x, Du) + \int_\Omega \Psi(x, u(x))dx \tag{2.3.1}$$

over the space $BV(\Omega)$. Henceforth, we assume that Φ is a Finsler integrand and that

(H9) $\Psi(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in x, strictly convex and coercive in t, that is to say

$$\lim_{t\to\pm\infty} \Psi(x,t) = +\infty,$$

and such that

$$\Psi(\cdot,0) \in L^1(\Omega), \tag{2.3.2}$$

$$\partial_t^- \Psi(\cdot,t) \in L^1(\Omega)\ \forall t \in \mathbb{R}. \tag{2.3.3}$$

Remark 2.3.8.(*i*) We recall that function $\Psi(x,\cdot)$ being convex for any $x \in \mathbb{R}^N$ it is therefore locally Lipschitz continuous on $\mathbb{R}$ (see [80] for instance). We therefore denote

$$\partial_t \Psi(x,t) := \partial_t^- \Psi(x,t)$$

the left derivative that exists at any $t \in \mathbb{R}$.

(*ii*) Clearly the energy (ROF) is a special case of (2.3.1) since it amounts to take $\Phi(x,p) = |p|$ and $\Psi(x,u(x)) = \frac{1}{2}(u(x)-g(x))^2$ for some $g \in L^2(\Omega)$ and bounded Ω. Observe that one can also consider a general data fidelity term of the form $\Psi(x,u(x)) = \frac{1}{q}(u(x)-g(x))^q$ for some $g \in L^q(\Omega)$ with $q > 1$.

(*iii*) Let us note that for $t > s$ and $x \in \Omega$,

$$\partial_t \Psi(x,s)(s-t) \le \Psi(x,s) - \Psi(x,t) \le \partial \Psi_t(x,t)(t-s) \tag{2.3.4}$$

which, in conjunction with (2.3.2), implies that

$$\Psi(\cdot,t) \in L^1(\Omega)\ \forall t \in \mathbb{R}.$$

For further reference, let us also remark that if $t_n \to t \in \mathbb{R}$ then for n large,

$$|\Psi(x,t_n) - \Psi(x,t)| \le \sup_{k\ge n} |t_k - t| \max(|\partial_t \Psi(x,t-1)|, |\partial_t \Psi(x,t+1)|)$$

hence in particular $\Psi(\cdot,t_n) \to \Psi(\cdot,t)$ in $L^1(\Omega)$.

(*iv*) We could have replaced assumption (2.3.3) in ($H9$) by

$$\Psi(\cdot,t) \ge \psi \in L^1(\Omega)\ \forall t \in \mathbb{R},$$

if one considered *local minimizers* of the ROF problem on an unbounded domain but this would lead us too far. See the beginning of [49, Section 5] for further details.

Applying the direct method, we get readily

Proposition 2.3.9. *Let Φ be a Finsler integrand and Ψ measurable in x and strictly convex in the second variable, then $\mathcal{E}$ has a unique minimizer u in the space $BV(\Omega)$.*

Proof. Consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ such that $\mathcal{E}(u_n) \to \inf_\Omega \mathcal{E}$. As $\mathcal{E}(u_n) \leq \mathcal{E}(0) < +\infty$, assumption $(H2)$ implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(\Omega)$. Then, Rellich's theorem asserts that up to extraction of a subsequence (still denoted $(u_n)_{n\in\mathbb{N}}$) it converges in $L^1(\Omega)$ and also pointwise to some $u \in BV(\Omega)$. By lower-semicontinuity of J_Φ and Fatou, we get

$$\mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n) = \inf_\Omega \mathcal{E}(u).$$

This proves the existence of a minimizer namely u. It is unique by strict convexity of $\mathcal{E}$. $\square$

Remark 2.3.10. It is also possible to reason in a slightly different way to avoid using Rellich's theorem in case $\Psi(x, u(x)) = \frac{1}{q}(u(x) - g(x))^q$. This way we also avoid the regularity assumption on $\partial\Omega$. See the proof we gave for Proposition 3.6.1 or also [55, 58].

2.3.3 The minimization problem for level sets

We assume, for the time being, that Φ is a Finsler integrand and Ψ is as above. Let us introduce the following minimal surface problems parametrized by $t \in \mathbb{R}$

$$\min_E P_\Phi(E, \Omega) + \int_E \partial_t \Psi(x, t)dx. \tag{2.3.5}$$

The minimization is carried out on all sets of finite perimeter. Simply reasoning as in the previous proof we get the existence of minimizers (and again, in some cases, it is possible to avoid using Rellich's theorem as was done in [58]). Obviously, we may not have a unique solution. Given $t \in \mathbb{R}$, we shall denote E_t a solution of (2.3.5).

The following comparison result similar to [53, Lemma 2.1] and [5, Lemma 4] will be needed:

Lemma 2.3.11. *Let $f_1, f_2 \in L^1(\Omega)$ and E, F be respectively minimizers of*

$$\min_E P_\Phi(E, \Omega) - \int_E f_1(x)dx \quad \textit{and} \quad \min_F P_\Phi(F, \Omega) - \int_F f_2(x)dx$$

Then, if $f_1 < f_2$ a.e., $|E \setminus F| = 0$ (i.e. $E \subset F$ up to a negligible set).

Proof. First, observe that by the triangle inequality

$$J_\Phi(\chi_E + \chi_F, \Omega) \leq P_\Phi(E, \Omega) + P_\Phi(F, \Omega).$$

Whereas, by the coarea formula we also know that

$$J_\Phi(\chi_E + \chi_F, \Omega) = \int_0^2 P_\Phi(\{\chi_E + \chi_F > t\}, \Omega)dt = P_\Phi(E \cup F, \Omega) + P_\Phi(E \cap F, \Omega).$$

This proves that

$$P_\Phi(E \cap F, \Omega) + P_\Phi(E \cup F, \Omega) \leq P_\Phi(E, \Omega) + P_\Phi(F, \Omega). \tag{2.3.6}$$

Now, by minimality of E and F, we get

$$\begin{aligned} P_\Phi(E, \Omega) - \int_E f_1(x)dx &\leq P_\Phi(E \cap F, \Omega) - \int_{E\cap F} f_1(x)dx, \\ P_\Phi(F, \Omega) - \int_F f_2(x)dx &\leq P_\Phi(E \cup F, \Omega) - \int_{E\cup F} f_2(x)dx. \end{aligned}$$

Adding both inequalities and using (2.3.6), we have

$$\int_{E\setminus F} (f_1(x) - f_2(x))\, dx \geq 0$$

hence the result since $f_1 < f_2$ a.e.

□

In particular, we observe that

Lemma 2.3.12. *If $t < t'$ and E_t, $E_{t'}$ are the corresponding minimizers of the minimal surface problem (2.3.5) then $E_{t'} \subset E_t$ up to a negligible set.*

Proof. Note that the strict convexity implies $\partial_t\Psi(\cdot, t) < \partial_t\Psi(\cdot, t')$ thus the statement follows from the previous lemma.

□

Knowing this we can, as in [55], introduce

$$E_t^- = \bigcup_{t'>t} E_{t'},\ E_t^+ = \bigcap_{t'<t} E_{t'},$$

respectively the smallest solution and largest solution of

$$\min_E P_\Phi(E, \Omega) + \int_E \partial_t\Psi(x, t)dx.$$

One has to be careful because the sets $E_{t'}$ are defined up to negligible sets. As a consequence, the non-countable union and intersection may not be well-defined.

To remedy this problem one could have taken as a representative for $E_{t'}$ the set of points of density 1 which is also an open set. This can be shown thanks to the density lemma for the anisotropic perimeter (see [45, 60]). We will come back to this later (see Lemma 2.4.13 and the remark that follows).

If we set

$$v(x) := \sup\{t \in \mathbb{R} \ / \ x \in E_t\}.$$

It is easily seen that $\{v > t\} = E_t^-$ and that $\{v \geq t\} = E_t^+$.

Proceeding as for the total variation (see [53]), we get

Lemma 2.3.13. *Let Φ be a reversible Finsler integrand and Ψ as in $(H9)$. Then v is the minimizer of $\mathcal{E}$.*

Proof. Step 1. We claim that $\Psi(\cdot, v) \in L^1(\Omega)$. Let us prove it. Since E_t^- solves (2.3.5), we have in particular

$$P_\Phi(E_t^-, \Omega) + \int_{E_t^-} \partial_t \Psi(x,t)dx \leq 0. \tag{2.3.7}$$

Integrating with respect to t it follows

$$\int_0^M \int_{E_t^-} \partial_t \Psi(x,t)dxdt \leq 0$$

where by Fubini's theorem the integral to the left can be rewritten

$$\begin{aligned}\int_0^M \int_{E_t^-} \partial_t \Psi(x,t)dxdt &= \int_{E_0^-} \int_0^{\min(v(x),M)} \partial_t \Psi(x,t)dtdx \\ &= \int_{E_0^-} \big(\Psi(x, \min(v(x), M)) - \Psi(x,0)\big)dx\end{aligned}$$

which implies

$$\int_{\{v>0\}} \Psi(x, \min(v(x), M))dx \leq \int_\Omega \Psi(x,0) < +\infty.$$

Now, by Fatou's lemma, that can by applied by Remark 2.3.8,

$$\int_{\{v>0\}} \Psi(x, v(x))dx < +\infty.$$

Observe that by Remark 2.3.5 (this is where the reversibility of Φ comes into play), $\{-v > t\} = \{v < -t\} = \Omega \setminus E^+_{-t}$ solves

$$\min_E P_\Phi(E, \Omega) - \int_E \partial_t \Psi(x, -t) dx.$$

Thus replacing $\Psi(\cdot, t)$ by $\Psi(\cdot, -t)$ in (2.3.5), function v is replaced by $-v$. This proves

$$\int_{\{v<0\}} \Psi(x, v(x)) dx < +\infty$$

hence our claim.

Step 2. Let $v' \in BV(\Omega)$ such that $\Psi(\cdot, v') \in L^1(\Omega)$ a candidate for the minimization and denote $E'_t = \{v' > t\}$ for some $t \in \mathbb{R}$. By minimality of E^-_t,

$$P_\Phi(E^-_t, \Omega) + \int_{E^-_t} \partial_t \Psi(x, t) dx \le P_\Phi(E'_t, \Omega) + \int_{E'_t} \partial_t \Psi(x, t) dx.$$

Integrating with respect to t

$$\int_{-M}^{M} \left(P_\Phi(E^-_t, \Omega) + \int_{E^-_t} \partial_t \Psi(x, t) dx \right) dt \le \int_{-M}^{M} \left(P_\Phi(E'_t, \Omega) + \int_{E'_t} \partial_t \Psi(x, t) dx \right) dt. \tag{2.3.8}$$

Note that by Fubini's theorem

$$\begin{aligned} \int_{-M}^{M} \int_{E^-_t} \partial_t \Psi(x, t) dx dt &= \int_\Omega \int_{-M}^{M} \chi_{\{v>t\}} \partial_t \Psi(x, t) dt dx \\ &= \int_\Omega \int_{\min(v(x), -M)}^{\min(v(x), M)} \partial_t \Psi(x, t) dt dx \\ &= \int_\Omega \Psi(x, \min(v(x), M)) - \Psi(x, \min(v(x), -M)) dx \\ &= \int_\Omega \Psi(x, v(x)) dx - \int_\Omega \Psi(x, -M) dx + \mathcal{R}(v, M) \end{aligned}$$

where we set

$$\begin{aligned} \mathcal{R}(v, M) = &\int_\Omega \Psi(x, \min(v(x), M)) - \Psi(x, v(x)) dx \\ &+ \int_\Omega \Psi(x, -M) - \Psi(x, \min(v(x), -M)) dx. \end{aligned}$$

For the function v' one obtains a similar identity, namely

$$\int_{-M}^{M}\int_{E'_t} \partial_t\Psi(x,t)dxdt = \int_\Omega \Psi(x,v'(x))dx - \int_\Omega \Psi(x,-M)dx + \mathcal{R}(v',M).$$

Though, for any function v such that $\Psi(\cdot,v) \in L^1$

$$\lim_{M\to+\infty} \mathcal{R}(v,M) = 0.$$

Indeed, observe that on $\{v > M\}$

$$|\Psi(\cdot,\min(v,M)) - \Psi(\cdot,v)| = \Psi(\cdot,v) - \Psi(\cdot,M) \le \Psi(\cdot,v),$$

and on $\{v < -M\}$

$$|\Psi(\cdot,-M) - \Psi(\cdot,\min(v,-M))| = \Psi(\cdot,v) - \Psi(\cdot,-M) \le \Psi(\cdot,v),$$

which proves the claim by application of the dominated convergence theorem.

In the end, making $M \to +\infty$ in (2.3.8) and using the anisotropic coarea formula we get

$$\int_\Omega \Phi(x,Dv) + \int_\Omega \Psi(x,v(x))dx \le \int_\Omega \Phi(x,Dv') + \int_\Omega \Psi(x,v'(x))dx$$

which means that v minimizes $\mathcal{E}$ that is $v = u$ since the minimizer is unique.

□

As a consequence, one actually proved

Proposition 2.3.14. *Let Φ be a reversible Finsler integrand, Ψ satisfy $(H9)$ and u be the minimizer of $\mathcal{E}$. Then the superlevel $E_t := \{u > t\}$, $t \in \mathbb{R}$, solves the minimal surface problem*

$$\min_E P_\Phi(E,\Omega) + \int_E \partial_t\Psi(x,t)dx$$

over all sets of finite perimeter in Ω.

Remark 2.3.15. The case $\Psi(x,t) = F(t - g(x))$ with F of class C^1 and merely convex is discussed in [150]. The proof is based on an approximation argument. We expect the argument to work for a general integrand $\Psi(x,t)$ convex in t. Though, for our future analysis, such a refinement is not necessary.

2.4 The discontinuity set

In this section, we are interested in proving qualitative results on the behavior of the jump set of the minimizer of (2.3.1). For this purpose, we first need to deal with the regularity of the level sets of the minimizer.

2.4.1 Regularity theory for elliptic PDEs

Let us recall a classical result that is taken from Gilbarg and Trudinger's book [89] (see in particular Theorem 8.9 and Theorem 9.15). First, let us consider an operator in non divergence form

$$L = \sum_{i,j} a_{i,j} \partial_{x_i x_j} + \sum_i b_i \partial_{x_i} + c,$$

that satisfies the ellipticity condition

$$\sum_{i,j} a_{i,j} \xi_i \xi_j \geq C|\xi|^2.$$

We will say that L is *strictly elliptic*.

For such an operator, one has a general existence and regularity result for the Dirichlet problem:

Theorem 2.4.1 ([89, Theorem 9.15])**.** *Let Ω be a $C^{1,1}$ open domain in $\mathbb{R}^N$, and let the operator L be strictly elliptic in Ω with coefficients $a_{i,j} \in C^0(\bar{\Omega})$, b_i, $c \in L^\infty(\Omega)$, with i, $j = 1, \cdots, n$ and $c \leq 0$. Then if $f \in L^p(\Omega)$ and $\varphi \in W^{2,p}(\Omega)$ with $1 < p < \infty$, the Dirichlet problem $Lu = f$ in Ω, $u - \varphi \in W_0^{1,p}$ has a unique solution $u \in W^{2,p}(\Omega)$.*

Here me must mention the names of Ennio De Giorgi, John Nash and Jürgen Kurt Moser whose pioneering works contributed the most to the theory of regularity of elliptic PDEs. We refer to the survey of [122] for further PDE related regularity results and historical facts.

Remark 2.4.2. Let us also recall that for Sobolev spaces we have the following embedding in Hölder spaces

$$W^{k,p}(\Omega) \subset C^{r,\alpha}(\Omega)$$

when $k - r - \alpha = \frac{N}{p}$ and $\alpha \in (0, 1)$.

This is a consequence of Morrey's inequality (see [83]). Consequently, if $p > N$ in the previous theorem, the solution inherits more regularity, namely $u \in C^{1,\alpha}$ with $\alpha = 1 - \frac{p}{N}$.

2.4.2 Regularity issues for minimal surfaces

The classical regularity theory for minimal surfaces (see [144, 143, 31]) and the recent paper [79, Theorem 6.1], which discusses the regularity of rectifiable currents that are almost minimizers of an elliptic integrand, yield

Theorem 2.4.3. *Let Φ be any strongly convex Finsler integrand and Ψ be such that assumption $(H9)$ is satisfied. We also assume that for some real t, $\partial_t\Psi(\cdot,t) \in L^p(\Omega)$ with $p > N$. Then a set E_t that solves (2.3.5) has a reduced boundary $\partial^* E_t$ of Hölder class $C^{1,\alpha}$ for any $\alpha < \frac{1}{2}\left(1 - \frac{N}{p}\right)$.*
Moreover, $\partial E \setminus \partial^* E$ is a closed set and

$$\mathcal{H}^s(\partial E_t \setminus \partial^* E_t) = 0$$

for every $s > N - 3$.

The hypothesis $\partial_t\Psi(\cdot,t) \in L^p(\Omega)$ with $p > N$ is essential. Indeed, in [20], the authors even prove that any set of finite perimeter $E \subset \mathbb{R}^N$ solves the prescribed mean curvature problem

$$\min_E P(E, \mathbb{R}^N) + \int_E f$$

for some appropriate $f \in L^1(\mathbb{R}^N)$.

Morgan proved in [126] that the value $N-3$ is sharp by exhibiting an example of a singular Φ-minimizing hypersurface in $\mathbb{R}^4$.

When the anisotropy takes the form $\Phi(x,p) = w(x)|p|$ for some Hölder continuous weight w, it is possible to refine these regularity results and get $N - 8$ instead of $N - 3$ without even using the theory of currents. We shall discuss this case with many more details and references in Section 2.4.4.

If one assumes in addition that $\partial_t\Psi(\cdot,t) \in L^\infty$ we can actually gain a little more regularity:

Theorem 2.4.4. *Let Φ be a strongly convex Finsler integrand that is Lipschitz continuous in p uniformly x and consider a function Ψ that satisfies $(H9)$. Suppose that for some $t \in \mathbb{R}$, $\partial_t\Psi(\cdot,t) \in L^\infty$ and pick E_t that solves (2.3.5). Then $\partial^* E_t$ is $W^{2,p}$ for all $1 < p < \infty$ and thus $C^{1,\alpha}$ for any $\alpha < 1$.*
In addition, $\partial E \setminus \partial^* E$ is closed and

$$\mathcal{H}^s(\partial E \setminus \partial^* E) = 0$$

for every $s > N - 3$.

We actually improve Theorem 2.4.3 since the degree of Hölder continuity of the boundary increases from $\alpha/2$ to α. This result is stated in [7, p.140] for the classical curvature problem (2.2.3). As we could not find any precise reference for this more general case, we provide a proof.

First, let us point out that $\partial^* E_t$ is $W^{2,2}_{loc}$ as a consequence of the following lemma:

Lemma 2.4.5. *Let $v \in C^1(\Omega')$ with $\Omega' \subset \mathbb{R}^M$ open be a weak solution of*

$$-\operatorname{div}(A(\cdot, v, \nabla v)) = h \tag{2.4.1}$$

where $h \in L^\infty(\Omega')$ and $A : \Omega' \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}^M$ is Lipschitz continuous and locally strictly monotone i.e. *for any compact set $K \subset \mathbb{R}^M$ there is a constant c_K s.t.*

$$\langle A(x,t,p) - A(x,t,\tilde{p}), p - \tilde{p} \rangle \geq c_K |p - \tilde{p}|^2, \ \forall p, \tilde{p} \in K \tag{2.4.2}$$

uniformly $x \in \Omega', t \in \mathbb{R}$. Then $v \in W^{2,2}_{loc}(\Omega')$.

Remark 2.4.6. Notice that the mean curvature equation

$$\operatorname{div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) = h \tag{2.4.3}$$

is a special case of (2.4.1) corresponding to

$$A(x,t,p) = \frac{p}{\sqrt{1+|p|^2}}.$$

The latter does satisfy the ellipticity condition (2.4.2).

The proof of the lemma is based on Nirenberg's method (see [42] for instance for further details). We simply adapt the proof given in [8, Proposition 7.56].

Proof. Since the property we are interested is local, we can assume that Ω' is a ball of measure less than 1, that $\|\nabla v\|_\infty \leq M_v$ for some positive M_v. We will consider that $K = \overline{B(0, M_v)}$ and will denote respectively L_v, L_A the Lipschitz constants of v, A in K and M_A the maximum of A over K. For any generic function u we denote the difference

$$\Delta_\varepsilon u(x) = \frac{u(x + \varepsilon e_i) - u(x)}{\varepsilon}$$

in the direction e_i, i ranging from 1 to M.

Now, in the weak formulation of (2.4.1), we take as test functions $\varphi(\cdot - \varepsilon e_i)$ and φ with $\varphi \in C_c^\infty(\Omega')$, $\varepsilon > 0$ small enough and substract the two identities to get after a change of variable

$$\frac{1}{\varepsilon}\int_{\Omega'} \langle A(x+\varepsilon e_i, v(x+\varepsilon e_i), \nabla v(x+\varepsilon e_i)) - A(x, v(x), \nabla v(x)), \nabla\varphi(x)\rangle dx$$
$$= -\int_{\Omega'} h\Delta_{-\varepsilon}\varphi$$

which can be rewritten as

$$\frac{1}{\varepsilon}\int_{\Omega'} \langle [A(x, v(x), \nabla v(x+\varepsilon e_i)) - A(x, v(x), \nabla v(x))], \nabla\varphi(x)\rangle dx$$
$$= -\frac{1}{\varepsilon}\int_{\Omega'} \langle [A(x+\varepsilon e_i, v(x+\varepsilon e_i), \nabla v(x+\varepsilon e_i)) - A(x, v(x), \nabla v(x+\varepsilon e_i))], \nabla\varphi(x)\rangle dx$$
$$-\int_{\Omega'} h\Delta_{-\varepsilon}\varphi.$$

We take $\varphi = \eta^2\Delta_\varepsilon v$ with $\eta \in C_c^1(\Omega')$ and $0 \leq \eta \leq 1$ a cut-off function. Notice that $\nabla\varphi = 2\eta\nabla\eta\Delta_\varepsilon v + \eta^2\Delta_\varepsilon(\nabla v)$ so using (2.4.2) we may estimate the first integral from below by

$$c_K\int_{\Omega'} \eta^2|\Delta_\varepsilon(\nabla v)|^2 - 2L_A M_v\|\nabla\eta\|_\infty \int_{\Omega'} \eta|\Delta_\varepsilon(\nabla v)|.$$

The second integral can be controlled by

$$L_A\sqrt{1+M_v^2}\left(2M_v\|\nabla\eta\|_\infty + \int_{\Omega'} \eta^2|\Delta_\varepsilon(\nabla v)|\right).$$

As for the last integral, we get the following bound from above

$$\|h\|_\infty\left(6M_v\|\nabla\eta\|_\infty + \int_{\Omega'} \eta^2|\Delta_\varepsilon(\partial_{x_i} v)|\right)$$

exactly as in the proof of [8, Proposition 7.56].

All in all, we get a uniform bound for

$$\int_{\Omega'} \eta^2|\Delta_\varepsilon(\nabla v)|^2$$

when $\varepsilon \to 0$. Though we already know that $\Delta_\varepsilon(\nabla v)$ converges in the sense of distributions to $\partial_{x_i}(\nabla v)$ so we obtain that it must have a L^2_{loc} representative in Ω'. $\square$

With the previous lemma in hands we can now turn to the proof of Theorem 2.4.4:

Proof of Theorem 2.4.4. As will be detailed in the proof of Theorem 2.4.8 (see especially equation (2.4.4)) the level sets ∂E_t can be locally represented as the graph of a C^1 function v that satisfies the following Euler-Lagrange equation

$$\operatorname{div}_{x'}\big(\nabla_{p'}\Phi(\cdot, v, -\nabla_{x'}v, 1)\big) + \partial_{x_N}\Phi(\cdot, v, -\nabla v, 1) = \partial_t\Psi\left((\cdot, v), t\right)$$

over a ball $B' \subset \mathbb{R}^{N-1}$ and note that we used the notation $x = (x', x_N)$ and $p = (p', p_N)$.

This actually means that function v solves in a weak sense

$$-\operatorname{div}_{x'}\big(\nabla_{p'}\Phi(\cdot, v, -\nabla_{x'}v, 1)\big) = h$$

for some $h \in L^\infty(B')$.

Now, $\nabla_p\Phi$ being Lipschitz continuous (see Remark 2.3.1), we can apply Lemma 2.4.5 so v is in $W^{2,2}_{loc}(B')$ and we are allowed to expand the divergence. Doing so, we find that v satisfies in a weak sense the following identity

$$\begin{aligned}-(\operatorname{div}_{x'}\nabla_{p'}\Phi)(\cdot, v, -\nabla_{x'}v, 1) - \nabla_{x'}v\cdot(\partial_{x_N}\nabla_{p'}\Phi)(\cdot, v, -\nabla_{x'}v, 1)&\\ + \operatorname{tr}\Big(D^2_{x'}v^T D^2_{p'}\Phi(\cdot, v, -\nabla_{x'}v, 1)\Big) &= h\end{aligned}$$

which can be rewritten as

$$\sum_{i,j=1}^{N-1} a_{i,j}\partial_{x_i x_j}v = \tilde{h}$$

with

$$\begin{aligned}a_{i,j} =&\partial_{p_i p_j}\Phi(\cdot, v, -\nabla_{x'}v, 1),\\ \tilde{h} =&h + (\operatorname{div}_{x'}\nabla_{p'}\Phi)(\cdot, v, -\nabla_{x'}v, 1)\\ &+ \nabla_{x'}v\cdot(\partial_{x_N}\nabla_{p'}\Phi)(\cdot, v, -\nabla_{x'}v, 1) \in L^\infty(B').\end{aligned}$$

The $W^{2,p}$ regularity for any $1 < p < \infty$ follows from well-known results on the regularity of solutions of elliptic partial differential equations in general form with continuous coefficients (see Theorem 2.4.1). Then Morrey's inequality (see Remark 2.4.2) yields the $C^{1,\alpha}$ regularity for any $\alpha < 1$. $\square$

Remark 2.4.7. Clearly the regularity theorem for elliptic equations in divergence form cannot be applied since the coefficients lack regularity. This is the reason why we proceeded by first proving Lemma 2.4.5 to be able to differentiate $\nabla_{x'}v$ and use the second regularity theorem for PDEs in non-divergence form.

2.4.3 The discontinuities of solutions of the anisotropic minimum problem

We are now ready to state the main result of this section:

Theorem 2.4.8. *Let Φ be a strongly convex reversible Finsler integrand of class C^2 on $\Omega \times \mathbb{R}^N \setminus \{0\}$, Ψ be as in (H9) and that satisfies in addition for some countable D dense in $\mathbb{R}$*

$$\partial_t \Psi(\cdot, t) \in BV(\Omega) \cap L^\infty(\Omega) \ \forall t \in D.$$

If $u \in BV(\Omega)$ is the minimizer of (2.3.1), then one has

$$J_u \subset \bigcup_{t \in D} J_{\partial_t \Psi(\cdot,t)}$$

up to a $\mathcal{H}^{N-1}$-negligible set.

Remark 2.4.9.(i) When Φ does not depend on x and if we set $\Psi(x,t)$ to get the classical quadratic data fidelity term the result was already stated in [48] and is the key step to get an extension of this theorem when dealing with TV_ε. This is not trivial since the latter functional is not positively 1-homogeneous. Let us denote u_λ the minimizer of

$$\min_{u \in BV(\Omega)} \lambda \int_\Omega \sqrt{1 + |Du|^2} + \frac{1}{2}\|u - g\|_2^2$$

where without loss of generality we dropped the ε.

The trick is to add another dimension and consider the functions

$$\tilde{u}(x, x_{N+1}) := u(x) + x_{N+1},$$
$$\tilde{g}(x, x_{N+1}) := g(x) + x_{N+1}.$$

Then it is possible to prove that $\tilde{u}_\lambda$ minimizes (locally)

$$\lambda \int_{\Omega \times \mathbb{R} \subset \mathbb{R}^{N+1}} |D\tilde{u}| + \frac{1}{2}\|\tilde{u} - \tilde{g}\|_2^2$$

thus $J_{u_\lambda} \times \mathbb{R} = J_{\tilde{u}_\lambda} \subset J_{\tilde{g}} = J_g \times \mathbb{R}$.

(ii) The ellipticity assumption for Φ is necessary. Indeed, in [24], it is shown that, in the crystalline case $\Phi(p) = \|p\|_1$ in dimension $N = 2$, it is possible to construct a function g such that one has for the corresponding minimizer u, $J_g \subsetneq J_u$.

Our proof follows closely the one given by Caselles, Chambolle and Novaga in [48].

Proof. Since for any countable set D dense in $\mathbb{R}$

$$J_u \subset \bigcup_{\substack{t_1,t_2\in D\\ t_1<t_2}} \partial E_{t_1} \cap \partial E_{t_2},$$

it is sufficient to prove that for all t_1, $t_2 \in D$

$$\partial E_{t_1} \cap \partial E_{t_2} \subset J_{\partial_t \Psi(\cdot,t_1)} \cup J_{\partial_t \Psi(\cdot,t_2)}$$

up to a $\mathcal{H}^{N-1}$-negligible set.

To prove the latter inclusion, we are going to reason by contradiction and assume that there are $t_1 \neq t_2$ such that

$$\mathcal{H}^{N-1}\left((\partial E_{t_1} \cap \partial E_{t_2}) \setminus (J_{\partial_t \Psi(\cdot,t_1)} \cup J_{\partial_t \Psi(\cdot,t_2)})\right) > 0.$$

Given that

$$\mathcal{H}^{N-1}\left((S_{\partial_t \Psi(\cdot,t_1)} \cup S_{\partial_t \Psi(\cdot,t_2)}) \setminus (J_{\partial_t \Psi(\cdot,t_1)} \cup J_{\partial_t \Psi(\cdot,t_2)})\right) = 0,$$

where we recall that $\Omega \setminus S_{\partial_t \Psi(\cdot,t_i)}$ is the set of approximate continuity points of $\partial_t \Psi(\cdot, t_i)$ (see Definition 2.2.7), it is equivalent to assume that

$$\mathcal{H}^{N-1}\left((\partial E_{t_1} \cap \partial E_{t_2}) \setminus (S_{\partial_t \Psi(\cdot,t_1)} \cup S_{\partial_t \Psi(\cdot,t_2)})\right) > 0.$$

By Theorem 2.4.3, one can get rid of the closed set where the boundary ∂E_{t_1} and ∂E_{t_2} are not regular and place ourself at a point

$$\bar{x} \in \partial^* E_{t_1} \cap \partial^* E_{t_2} \setminus (S_{\partial_t \Psi(\cdot,t_1)} \cup S_{\partial_t \Psi(\cdot,t_2)})$$

such that both these boundaries can be represented as graphs in the vicinity of $\bar{x}$. That is to say, up to a Euclidian motion, there is a cylindrical neighborhood $\{(x', x_N) \in \mathbb{R}^N \ / \ |x'| < R, \ -R < x_N < R\}$ of $\bar{x} = (\bar{x}', \bar{x}_N)$ for some small $R > 0$ such that $E_{t_i}, i \in \{1, 2\}$ coincides with the epigraph of a function $v_i : B(\bar{x}', R) \to (-R, R)$ of class $W^{2,q}$, for any $q \geq 1$. Again, throwing away $\mathcal{H}^{N-1}$-negligible sets one can assume that $\bar{x}'$ is a Lebesgue point of v_i, $\nabla_{x'} v_i$ and $D^2_{x'} v_i$. Actually, by Rademacher-Calderón's theorem, we know that v_i and $\nabla_{x'} v_i$ are differentiable a.e. on B' but this stronger result will not be necessary in what follows.

By Proposition 2.3.14, we know that both superlevels $E_{t_i}, \ i \in \{1, 2\}$ solve the following

$$\min_E \int_{\partial^* E} \Phi(x, \nu_E) d\mathcal{H}^{N-1} + \int_E \partial_t \Psi(x, t_i) dx$$

where we minimize over all sets of finite perimeter in Ω. By doing compact modifications in the ball $B' = B(\bar{x}', R)$, one can see that the graph $v_i,\ i \in \{1,2\}$, minimizes

$$I(v) = \int_{B'} \Phi\left(x', v(x'), \frac{(-\nabla_{x'}v(x'),1)}{\sqrt{1+|\nabla_{x'}v(x')|^2}}\right)\sqrt{1+|\nabla_{x'}v(x')|^2}\,dx' + \int_{B'}\int_{v(x')}^{R} \partial_t\Psi\left((x',x_N),t_i\right)dx_N dx'.$$

This means that, for any positive perturbation $\varphi \in C_c^\infty(B')$ of the level set ∂E_{t_i} with $i \in \{1,2\}$,

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{I(v_i+\varepsilon\varphi)-I(v_i)}{\varepsilon} \geq 0.$$

On the other hand, if we denote $p' = (p_1, ..., p_{N-1})$,

$$\begin{aligned} I(v_i+\varepsilon\varphi) = I(v_i) + \varepsilon\int_{B'}\Big(&\partial_{x_N}\Phi\left(x', v_i(x'), -\nabla_{x'}v_i(x'),1\right)\varphi(x') \\ &- \nabla_{p'}\Phi\left(x', v_i(x'), -\nabla_{x'}v_i(x'),1\right)\cdot\nabla_{x'}\varphi(x') \\ &- \int_{v_i(x')}^{v_i(x')+\varepsilon\varphi(x')} \partial_t\Psi\left((x',x_N),t_i\right)dx_N\Big)dx' + o(\varepsilon). \end{aligned} \tag{2.4.4}$$

One should note that the partial differentiations are done in the new set of coordinates. If we integrate by parts, which is possible by Corollary 2.4.4, and given the slicing properties of BV functions (see [8]), one has for $\mathcal{H}^{N-1}$-a.e. $x' \in B'$

$$\begin{aligned} \partial_{x_N}\Phi(x,\nu_{E_{t_i}}(x)) + [\mathrm{div}_{x'}\nabla_{p'}\Phi](x,\nu_{E_{t_i}}(x)) + \nabla_{x'}v_i(x')\cdot[\partial_{x_N}\nabla_{p'}\Phi](x,\nu_{E_{t_i}}(x)) \\ - D^2_{x'}v_i(\bar{x}') : D^2_{p'}\Phi(x,\nu_{E_{t_i}}(x)) - \partial_t\Psi\,(x,t_i+0) \geq 0, \end{aligned}$$

where $D^2_{x'}v_i(x') : D^2_{p'}\Phi(x,\nu_{E_{t_i}}(x))$ designates the tensor contraction of the Hessians and is defined as follows

$$\begin{aligned} D^2_{x'}v_i(x') : D^2_{p'}\Phi(x,\nu_{E_{t_i}}(x)) &= \mathrm{tr}\left({D^2_{x'}v_i}^T D^2_{p'}\Phi(x,\nu_{E_{t_i}}(x))\right) \\ &= \sum_{k,l=1}^{N-1} \partial_{x_k x_l}v_i(x')\ \partial_{p_k p_l}\Phi(x,\nu_{E_{t_i}}(x)). \end{aligned}$$

Reasoning in the same way with $\varepsilon < 0$, one gets

$$\begin{aligned} \partial_{x_N}\Phi(x,\nu_{E_{t_i}}(x)) - [\mathrm{div}_{x'}\nabla_{p'}\Phi](x,\nu_{E_{t_i}}(x)) - \nabla_{x'}v_i(x')\cdot[\partial_{x_N}\nabla_{p'}\Phi](x,\nu_{E_{t_i}}(x)) \\ + D^2_{x'}v_i(\bar{x}') : D^2_{p'}\Phi(x,\nu_{E_{t_i}}(x)) - \partial_t\Psi\,(x,t_i-0) \leq 0. \end{aligned}$$

Now, without loss of generality, we can assume $t_2 > t_1$. By Lemma 2.3.12, which asserts that $v_2 \geq v_1$ a.e. on B', one has

$$\begin{aligned} v_1(\bar{x}') &= v_2(\bar{x}'), \\ \nabla_{x'} v_1(\bar{x}') &= \nabla_{x'} v_2(\bar{x}'), \\ D^2_{x'} v_1(\bar{x}') &\leq D^2_{x'} v_2(\bar{x}'). \end{aligned}$$

Since in addition we assumed $\bar{x} \notin S_{\partial_t \Psi(\cdot, t_i)}$ for $i \in \{1,2\}$, we find

$$\begin{aligned} \partial_{x_N} \Phi(\bar{x}, \nu_{E_{t_i}}(\bar{x})) + [\operatorname{div}_{x'} \nabla_{p'} \Phi](\bar{x}, \nu_{E_{t_i}}(\bar{x})) + \nabla_{x'} v_i(\bar{x}') \cdot \partial_{x_N} \nabla_{p'} \Phi(\bar{x}, \nu_{E_{t_i}}(\bar{x})) \\ - D^2_{x'} v_i(\bar{x}') : D^2_{p'} \Phi(\bar{x}, \nu_{E_{t_i}}(\bar{x})) - \partial_t \Psi(\bar{x}, t_i) = 0. \end{aligned} \tag{2.4.5}$$

Therefore, substracting the equations (2.4.5) we got for the two values of i and using the strict convexity assumption on Ψ (see ($H9$)), we are simply left with

$$D^2_{p'} \Phi(\bar{x}, \nu_E(\bar{x})) : \left(D^2_{x'} v_1(\bar{x}') - D^2_{x'} v_2(\bar{x}') \right) = \partial_t \Psi(\bar{x}, t_2) - \partial_t \Psi(\bar{x}, t_1) > 0.$$

Whereas, by non-negativity of Φ which is asserted by ($H8$), it follows (see [106, p. 218])

$$D^2_{p'} \Phi(\bar{x}, \nu_E(\bar{x})) : \left(D^2_{x'} v_1(\bar{x}') - D^2_{x'} v_2(\bar{x}') \right) \leq 0$$

hence the contradiction. □

2.4.4 Refinement for a weighted regularization

In this section, we are going to focus on the case when the integrand Φ is simply given by a weight against the total variation measure namely

$$\Phi(x,p) = w(x)|p|.$$

For simplicity, we consider that

$$\Psi(x,t) = \frac{1}{2}\|t - g\|_2^2.$$

This corresponds to the classical quadratic data fidelity term for some Lebesgue measurable function g. Function Φ is a strictly convex reversible Finsler integrand as soon as

(H10) $w : \Omega \to \mathbb{R}$ is positive, β-Hölder with $\beta \in (0,1]$ and there exists a positive real C_w such that $C_w^{-1} \leq w \leq C_w$.

Henceforth, w will satisfy this assumption.

To sum up, from now on, given w that satisfies $(H10)$, we are interested in the minimizer u of the following problem

$$\min_{u \in BV(\Omega)} \int_\Omega w|Du| + \frac{1}{2}\|u - g\|_2^2. \tag{2.4.6}$$

Its superlevels $E_t = \{u > t\}$ solve the minimal surface problem

$$\min_E P_w(E, \Omega) + \int_E (t - g(x))dx \tag{2.4.7}$$

over sets of finite perimeter in Ω, where we recall that

$$P_w(E, \Omega) = \int_{\partial^* E} w(x) d\mathcal{H}^{N-1}(x)$$

is the weighted perimeter.

All the results we developed for general Finsler integrands are still valid in this special case. In particular, the anisotropic coarea formula and the regularity Theorem 2.4.3 for quasi minimizers of the perimeter hold true.

More regularity in the weighted case

As already seen in the proof of Theorem 2.4.8, it is important to be able to say that the level sets of minimizers of problems involving the total variation are regular namely $C^{1,\alpha}$ for some $\alpha \in (0, 1/2)$. Such results stem from the theory of regularity of minimal surfaces and have become classical in the literature. We already mentioned the works [144, 143, 31, 79] that deal with minimizers or quasi-minimizers of the perimeter in the anisotropic setting. In particular, they establish the regularity of minimizers of the perimeter that have a prescribed curvature.

Similar regularity results have also been established for various constraints. Let us mention two recent examples. In [127], Frank Morgan proved such a regularity for isoperimetric surfaces. Indeed, he shows that an isoperimetric hypersurface of dimension at most six in a smooth Riemannian manifold is a smooth submanifold. If the metric is merely Lipschitz, then it is still $C^{1,\alpha}$ for any $\alpha > 1$.

In the recent article [86], Figalli and Maggi are led to consider a problem with both a constraint on the curvature and on the volume *i.e.*

$$\min_E \left\{ \int_{\partial^* E} \Phi(\nu_E) d\mathcal{H}^{N-1} + \int_E g \ / \ |E| = m \right\} \tag{2.4.8}$$

for a positively 1-homogeneous elliptic Φ, coercive g and for small m. In Appendix C of their work, they discuss the $C^{1,\alpha}$ regularity of a solution of (2.4.8) as a consequence of the regularity theory for quasi-minimizers of the anisotropic perimeter [79].

The papers we just cited make a wide usage of the language of currents and varifolds. Some other works ([115, 116, 117, 147, 148, 11]) are based on techniques that date back to the results of De Giorgi and deal with these regularity issues in the framework of sets of finite perimeter. To keep the exposition as clear as possible we prefer to adopt their approach to get a regularity theorem for the weighted total variation. This way, we can provide a quite simple proof based on these results and that does not make any reference to the theory of currents. The aim is to make this work accessible to a wide audience that is not familiar with such concepts. Our result is partially contained in the above-mentioned Theorem 2.4.3 for the anisotropic total variation which follows from [79] but to our knowledge there is no simple proof of it in the context of sets of finite perimeter.

First, let us recall the concept of quasi-minimizer:

Definition 2.4.10. Let E be a set of finite perimeter in Ω, w satisfy assumption $(H10)$, $\alpha \in (0,1)$ and $\Lambda \geq 0$. Then E is a (Λ,α)-*quasi-minimizer of the perimeter* P_w *in* Ω or simply *quasi-minimizer* if

$$P_w\left(E, B(x,r)\right) \leq P_w\left(F, B(x,r)\right) + \Lambda |E \Delta F|^{1+\frac{2\alpha-1}{N}} \tag{2.4.9}$$

for any ball $B(x,r) \subset\subset \Omega$ with $r>0$ and any $F \subset \Omega$ of finite perimeter such that $F\Delta E \subset\subset B(x,r)$.

Remark 2.4.11. We could have replaced (2.4.9) by the weaker condition

$$P_w\left(E, B(x,r)\right) \leq P_w\left(F, B(x,r)\right) + \Lambda r^{N-1+2\alpha} \tag{2.4.10}$$

but for simplicity we refer to [147, 148] where the author considers this definition.

The aim is to show that the following regularity for quasi-minimizers of the weighted perimeter holds:

Theorem 2.4.12. *Let Ω be an open set of $\mathbb{R}^N$, $N \geq 2$, $w : \Omega \to \mathbb{R}$ β-Hölder for some $\beta \in (0,1]$ and such that there is a positive real C_w with $C_w^{-1} \leq w \leq C_w$. Consider also $\alpha \in (0, \frac{1}{2})$, $\Lambda \geq 0$ and E a set of finite perimeter that is a (Λ,α)-minimizer of the perimeter P_w.*

Then, if we denote $\gamma = \min(\alpha, \frac{\beta}{2})$, the reduced boundary $\partial^* E$ is a $C^{1,\gamma}$-hypersurface and

$$\mathcal{H}^s(\partial E \setminus \partial^* E) = 0$$

for every $s > N-8$.

Moreover, let us assume that $(E_h)_{h\in\mathbb{R}}$ are (Λ,α)-minimizers of the perimeter P_w with E_h converging locally to some limit set E_∞ as $h \to +\infty$. If $x_h \in \partial E_h$ for every h and converges as $h \to +\infty$ to some $x_\infty \in \Omega$ then $x_\infty \in \partial E_\infty$. If, in addition, $x_\infty \in \partial^* E_\infty$ then there exists h_0 such that for $h \geq h_0$, $x_h \in \partial^* E_h$ and the unit outward normal to $\partial^* E_h$ at x_h converges to the unit outward normal to $\partial^* E_\infty$ at x_∞.

The theorem is well-known for quasi-minimizers of the classical perimeter (even with the weaker condition (2.4.10)) and follows from [148] whose work is based on earlier papers of Massari ([115, 116, 117]). Thus, to get the announced regularity, it is sufficient to prove that a quasi-minimizer of P_w is a quasi-minimizer of the classical perimeter P. The argument is based on a key ingredient: the density lemma. The latter result is well-known for problems involving the perimeter. The density lemma will also play an important role in Section 2.5 to prove that minimizers of ROF have large flat zones.

Lemma 2.4.13 (Density estimate)**.** *Let w be as in assumption $(H10)$, $\alpha \in (0,1)$, $\Lambda \geq 0$ and consider E a set of finite perimeter that is a (Λ,α)-quasi-minimizer of P_w. Then there exists a radius $r_0 > 0$ and $C > 0$ depending only on N and w such that for any point $x \in \Omega$,*

- *if $\forall r > 0,\ |E \cap B(x,r)| > 0$ then $\forall r < r_0,\ |E \cap B(x,r)| \geq \frac{w_N r^N}{2^N C_w^N}$,*
- *if $\forall r > 0,\ |B(x,r) \setminus E| > 0$ then $\forall r < r_0,\ |B(x,r) \setminus E| \geq \frac{w_N r^N}{2^N C_w^N}$.*

In particular, if $x \in \partial^* E$,

$$\forall r < r_0,\ \min\left(|E \cap B(x,r)|, |B(x,r) \setminus E|\right) \geq \frac{w_N r^N}{2^N C_w^N}.$$

Moreover, one has for the classical perimeter

$$C^{-1} r^{N-1} \leq P(E, B(x,r)) \leq C r^{N-1}.$$

Remark 2.4.14.(i) The assertion on the perimeter is sometimes referred to as the *Ahlfors regularity* of the boundary ∂E.

(ii) A variant of this lemma holds also for the anisotropic perimeter P_Φ (see [45] for instance).

In our problem, the key point is that (2.4.9) can be rewritten in terms of the classical perimeter in the following way:

$$\begin{aligned} P_w(E, B(x,r)) &= w(x) P(E, B(x,r)) + \int_{\partial^* E} (w(y) - w(x)) d\mathcal{H}^{N-1} \\ &\leq w(x) P(F, B(x,r)) + \int_{\partial^* F} (w(y) - w(x)) d\mathcal{H}^{N-1} + \Lambda |E \Delta F|^{1+\frac{2\alpha-1}{N}} \end{aligned}$$

But, since w is β-Hölder, the quasi-minimality condition becomes

$$(w(x) - \|w\|_{C^{0,\beta}} r^\beta) P(E, B(x,r)) \leq (w(x) + \|w\|_{C^{0,\beta}} r^\beta) P(F, B(x,r)) + \Lambda |E \Delta F|^{1+\frac{2\alpha-1}{N}}.$$

To alleviate notations, we are going to assume that w has Hölder norm $\|w\|_{C^{0,\beta}} = 1$ and we are going to write B_r for the ball $B(x,r)$. So for some small radius r with $r^\beta < C_w^{-1}$, we are simply left with

$$(w(x) - r^\beta) P(E, B_r) \leq (w(x) + r^\beta) P(F, B_r) + \Lambda |E \Delta F|^{1+\frac{2\alpha-1}{N}}. \tag{2.4.11}$$

Having this remark in mind, we can now get to the proof:

Proof. Let us prove the first item of the lemma. The idea is to compare the energy of E with that of $E \setminus B_r$. Let $f(r) = |E \cap B_r| > 0$ for all $r > 0$. We see that it is a non decreasing function thus differentiable almost everywhere and by the coarea formula one knows that $f'(r) = \mathcal{H}^{N-1}(E \cap \partial B_r)$ for a.e. r. Now by the isoperimetric inequality,

$$N\omega_N^{\frac{1}{N}} f(r)^{\frac{N-1}{N}} \leq P(E \cap B_r, \mathbb{R}^N) = \mathcal{H}^{N-1}(\partial^* E \cap B_r) + \mathcal{H}^{N-1}(E \cap \partial B_r).$$

But by minimality of E,

$$\mathcal{H}^{N-1}(\partial^* E \cap B_r) \leq \frac{w(x) + r^\beta}{w(x) - r^\beta} \mathcal{H}^{N-1}(E \cap \partial B_r) + \Lambda \frac{f(r)^{1+\frac{2\alpha-1}{N}}}{w(x) - r^\beta}.$$

So

$$\left(N\omega_N^{\frac{1}{N}} - \frac{\Lambda f(r)^{\frac{2\alpha}{N}}}{w(x) - r^\beta} \right) f(r)^{\frac{N-1}{N}} \leq \left(1 + \frac{w(x) + r^\beta}{w(x) - r^\beta} \right) f'(r),$$

which implies

$$\frac{w(x) - r^\beta}{2} \left(\omega_N^{\frac{1}{N}} - \frac{\Lambda f(r)^{\frac{2\alpha}{N}}}{N(w(x) - r^\beta)} \right) \leq \left(f(r)^{\frac{1}{N}} \right)'.$$

Now for $\varepsilon \in (0,1)$, we can find $r_\varepsilon > 0$ such that for a.e. $r < r_\varepsilon$,

$$\frac{w(x) - {r_\varepsilon}^\beta}{2} \left(\omega_N^{\frac{1}{N}} - \frac{\Lambda \varepsilon}{N(w(x) - {r_\varepsilon}^\beta)} \right) \leq \left(f(r)^{\frac{1}{N}} \right)'.$$

Integrating between 0 and r_ε and sending $\varepsilon \to 0$, one obtains

$$\frac{w(x)^N w_N}{2^N} \leq \liminf_{r \to 0} \frac{f(r)}{r^N}$$

hence the first assertion of the lemma.

Reasoning in a similar way with $f(r) = |B_r \setminus E|$ one proves the second item of the lemma.

Let us prove the last statement. As above by comparison of E with $E \setminus B_r$ one has

$$(w(x) - r^\beta)P(E, B_r) \leq (w(x) + r^\beta)\mathcal{H}^{N-1}(E \cap \partial B_r) + \Lambda |E \cap B_r|^{1-\frac{1}{N}}.$$

Thus we obtain

$$P(E, B_r) \leq C\left(\Lambda + \frac{C_w + r^\beta}{C_w^{-1} - r^\beta}\right) r^{N-1}$$

for some constant C that only depends on N.

In the last statement, the inequality to the left is obtained by applying the relative isoperimetric inequality. $\square$

With this lemma in hands, we are in a position to prove that a quasi-minimizer of the weighted perimeter is simply a quasi-minimizer of the perimeter. Take E a quasi-minimizer of P_w as in Definition 2.4.9. Then, using the notations introduced before the proof of the lemma, we have from (2.4.11) that for any admissible F,

$$\begin{aligned} P(F, B_r) + \Lambda \frac{|E \Delta F|^{1+\frac{2\alpha-1}{N}}}{w(x) + r^\beta} &\geq \frac{w(x) - r^\beta}{w(x) + r^\beta} P(E, B_r) \\ &\geq (1 - 2C_w r^\beta) P(E, B_r) \\ &\geq P(E, B_r) - 2CC_w r^{N-1+\beta}, \end{aligned}$$

where in the last line we used the density lemma. Then

$$\begin{aligned} P(E, B_r) &\leq P(F, B_r) + r^{N-1}(\Lambda C_w r^{2\alpha} + 2CC_w r^\beta) \\ &\leq P(F, B_r) + C' r^{N-1+2\gamma}, \end{aligned}$$

for some positive constant C' and Theorem 2.4.12 follows from the regularity of quasi-minimizers for the classical perimeter (see Remark 2.4.11 and also [147, 148]).

Consider now u a minimizer of (2.4.6) for $g \in L^p(\Omega)$ with $p > N$ and let $E_t = \{u > t\}$ for some $t \in \mathbb{R}$. Then, if $x \in \Omega$, $r > 0$ and F is a compact modification of E_t in $B(x,r)$ *i.e.* $F\Delta E_t \subset\subset B(x,r)$, one has

$$\begin{aligned} P_w(E_t, B(x,r)) &\leq P_w(F, B(x,r)) + \int_{E_t\Delta F} |t-g| \\ &\leq P_w(F, B(x,r)) + |E_t\Delta F|^{1-\frac{1}{p}} \|t-g\|_{L^p(B(x,r))} \\ &\leq P_w(F, B(x,r)) + \Lambda r^{N\left(1-\frac{1}{p}\right)}, \end{aligned}$$

so the superlevel E_t is a quasi-minimizer of P_w (and satisfies also the weaker definition (2.4.10)). Therefore, Theorem 2.4.12 applies with $\alpha = \frac{1}{2}(1-\frac{N}{p})$. In the end, we get exactly the same regularity as in Theorem 2.4.3 but this time we also know that the singular set has dimension at most $N-8$.

Note that there is no way to adapt our approach in the anisotropic case since this would contradict the counterexample of Frank Morgan (see the remark that follows Theorem 2.4.3). Thus, a quasi-minimizer of the anisotropic perimeter is not necessarily a quasi-minimizer of the classical perimeter.

Now if one assumes that $g \in L^\infty(\Omega)$, we recall that the Nirenberg's method and the regularity theory for elliptic PDEs let us gain a little regularity as can be seen from the following reformulation of Theorem 2.4.4:

Theorem 2.4.15. *Let Ω be an open set of $\mathbb{R}^N$, $N \geq 2$, $w : \Omega \to \mathbb{R}$ be Lipschitz continuous and such that there exists a positive real C_w with $C_w^{-1} \leq w \leq C_w$. Consider also E a set of finite perimeter that is a quasi-minimizer of the perimeter P_w. Then $\partial^* E$ is $W^{2,p}$ for all $1 < p < \infty$ and thus $C^{1,\gamma}$ for any $\gamma < 1$ and*

$$\mathcal{H}^s(\partial E \setminus \partial^* E) = 0$$

for every $s > N-8$.

Remark 2.4.16. As was done before (see (2.4.4)), the level set ∂E_t can be locally represented as the graph of a C^1 function v that satisfies the following Euler-Lagrange

$$-\operatorname{div}_{x'}\left(w(x',v(x'))\frac{\nabla_{x'}v(x')}{\sqrt{1+|\nabla_{x'}v(x')|^2}}\right) + \partial_{x_N}w(x',v(x'))\sqrt{1+|\nabla_{x'}v(x')|^2} = \big(t - g(x',v(x'))\big)$$

over a ball $B' \subset \mathbb{R}^{N-1}$. We recall that we denoted $x = (x', x_N) \in \mathbb{R}^N$. It follows that function v solves

$$-\operatorname{div}_{x'}\left(w(\cdot,v)\frac{\nabla_{x'}v}{\sqrt{1+|\nabla_{x'}v|^2}}\right)=h$$

with $h \in L^\infty(B')$. In particular when $N=2$, Rademacher's theorem implies that $w(\cdot,v)v'/\sqrt{1+|v'|^2}$ is Lipschitz continuous which in turn implies that v is locally of class $C^{1,1}$. This provides additional regularity for the weighted total variation in dimension 2.

Discontinuities for the adaptive total variation minimization problem

In the weighted case, we can now get a refinement of the jump inclusion result.

Theorem 2.4.17. *Let $w:\Omega\to\mathbb{R}$ be positive, bounded, Lipschitz continuous with $\nabla w \in BV(\Omega,\mathbb{R}^N)$ and $g \in BV(\Omega)\cap L^\infty(\Omega)$. Then, denoting $J_{\nabla w} := \bigcup_{i=1}^N J_{\partial_{x_i}} w$, the minimizer $u \in BV(\Omega)$ of (2.4.6) satisfies*

$$J_u \subset J_g \cup J_{\nabla w} \tag{2.4.12}$$

up to a $\mathcal{H}^{N-1}$-negligible set.

Moreover, we have the following bound on the mean curvature of the jump set

$$\kappa_{J_u} \in \left[C_w^{-1}(g^- - u^-), C_w(g^+ - u^+)\right] \ \mathcal{H}^{N-1}\text{-a.e.}$$

If in addition we assume that w is of class C^1 we get that at the discontinuity

$$(u^+ - u^-) \le (g^+ - g^-) \ \mathcal{H}^{N-1}\text{-a.e. on } J_u. \tag{2.4.13}$$

Remark 2.4.18. Assumption $\nabla w \in BV(\Omega,\mathbb{R}^N)$ means that w lies in the space $BH(\Omega)$ of bounded Hessian functions that has been thouroughly studied by Demengel in [75]. It is possible to obtain results that are similar in nature to those known for the BV space. Let us mention in particular that these functions have a $W^{1,1}$ trace, there is also an extension theorem, a Poincaré type inequality and a continuous inclusion $BH(\Omega) \subset C^0(\bar\Omega)$ for Lipschitz domains of $\mathbb{R}^2$ (see also [142]).

Let us make few comments before getting to the proof. First of all it is interesting to see that whenever the weight is merely Lipschitz continuous it is possible to add discontinuities to the minimizer that were not contained in the datum g.

To illustrate this point, we give few numerical experiments in dimension one:

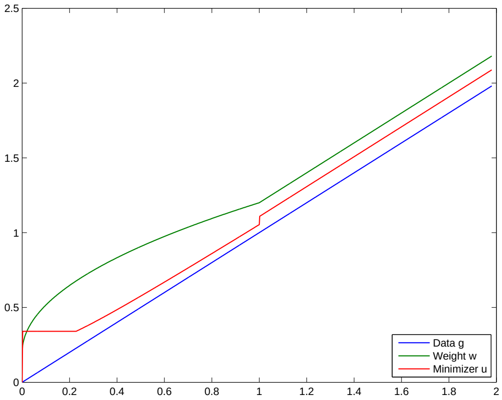


Figure 2.3: Creation of jumps with $w(x) = \sqrt{x}\chi_{\{x \leq 1\}} + x\chi_{\{x>1\}} + 0.2$

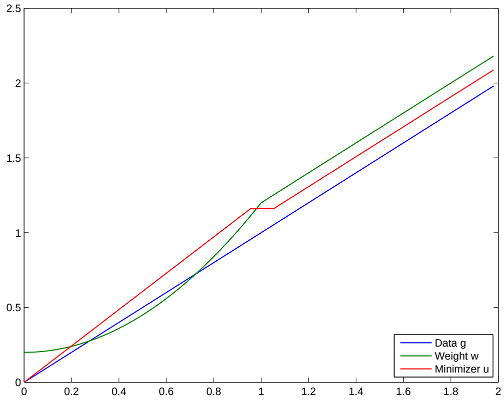


Figure 2.4: Creation of a flat zone for $w(x) = x^2\chi_{\{x \leq 1\}} + x\chi_{\{x>1\}} + 0.2$

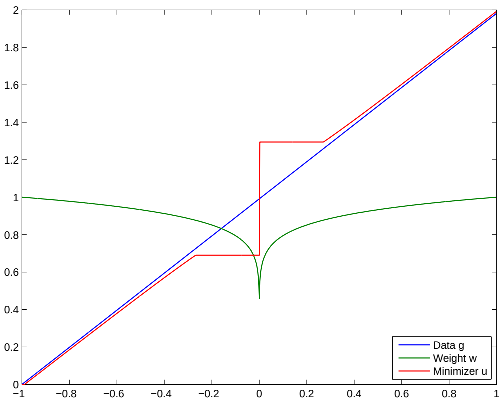


Figure 2.5: Hölder continuous weight $w(x) = |x|^{1/10}$

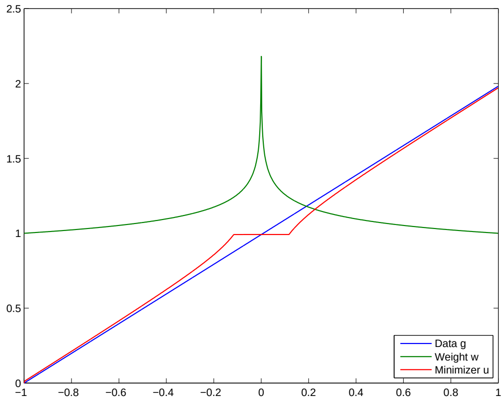


Figure 2.6: Weight function $w(x) = |x|^{-1/10}$

These simulations suggest that tweaking w so that w' contains a discontinuity one can obtain desired properties for u and for instance force the creation of jumps for some to be processed smooth g.

In case the weight function w is of class C^1, $J_{\nabla w} = \emptyset$ which implies merely $J_u \subset J_g$.

For such a smooth w, (2.4.13) means that the "contrast" (if one thinks of images) decreases at the discontinuity set J_u. This is not that surprising for natural images but quite counterintuitive if we consider the following function

$$\begin{array}{rccl} g: & [0,2\pi)^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \begin{cases} 2+\cos(x) \text{ if } y>0, \\ 0 \text{ otherwise.} \end{cases} \end{array}$$

provided periodic boundary conditions. Let us illustrate this example by a numerical experiment to get a clear idea of what is going on: we minimize ROF functional with the data function g that is above.

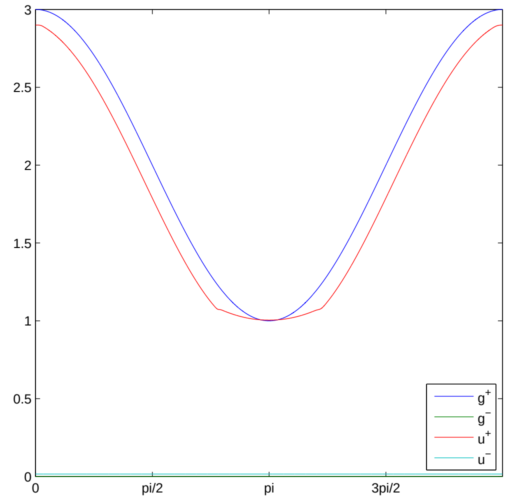


Figure 2.7: Functions u and g at the jump set.

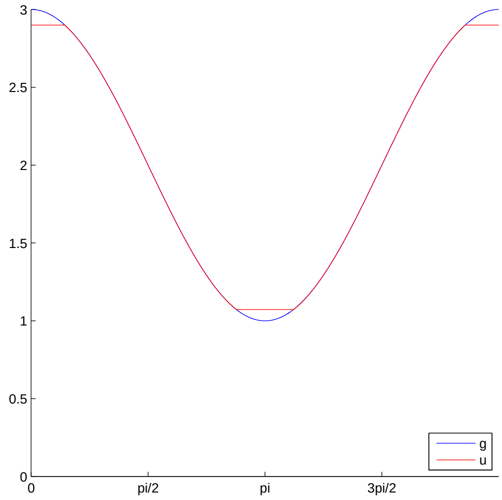


Figure 2.8: Functions u and g far from the discontinuity.

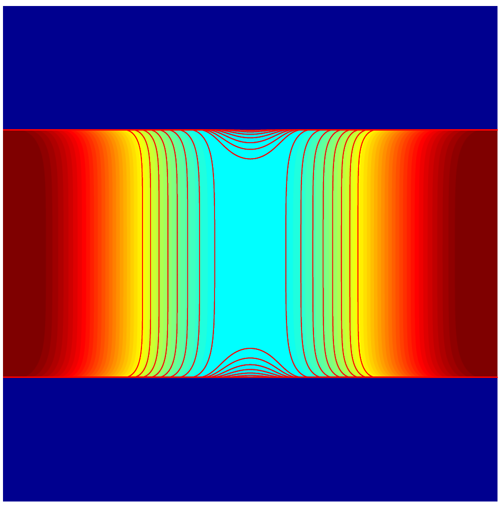


Figure 2.9: Level lines $\{u = t\}$ for some values of $t \in (1, 2)$.

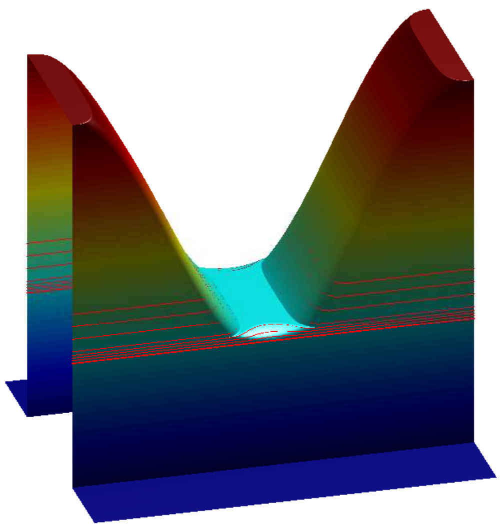


Figure 2.10: Graph of u on one period. Some level lines are represented in red.

One can clearly see that little bumps are created near the discontinuities to keep the jump as small as possible. We recall that, far from the jump set, we expect the solution to be constant on large neighborhoods of the extrema and to have a lower infinity norm. This is the object of the forthcoming Section 2.5 (see in particular Theorems 2.5.1 and 2.5.3).

The proof that follows is slightly different from the one given for Theorem 2.4.8 since this time we are no longer going to reason by contradiction. This way we can get the desired refinement in the weighted case.

Proof. We recall that up to a $\mathcal{H}^{N-1}$-negligible set

$$J_u \subset \bigcup_{\substack{t_1,t_2\in D\\ t_1<t_2}} \partial E_{t_1} \cap \partial E_{t_2}$$

for any countable D dense in $\mathbb{R}$, thus it is enough that the result for any $t_1, t_2 \in D$ and for $\mathcal{H}^{N-1}$-any $\bar{x} \in \partial E_{t_1} \cap \partial E_{t_2}$. Combining Theorems 2.4.12 and 2.4.15 one can assume that both these boundaries can be represented by smooth graphs near $\mathcal{H}^{N-1}$-every $\bar{x}$. That is to say that up to a Euclidian motion there exists a cylinder $\{x = (x', x_N) \in \mathbb{R}^N \ / \ |x'| < R,\ -R < x_N < R\}$ neighborhood of $\bar{x}$ such that E_{t_i}, $i \in \{1, 2\}$, coincides with the epigraph of a function $v_i : B' = B(\bar{x}', R) \to (-R, R)$ of class $W^{2,q}$ for any $q \geq 1$. We also assume that we have

$$\mathcal{H}^{N-1}\big(\{x' \in B' \ / \ v_1(x') = v_2(x')\}\big) > 0$$

for the contact set. Without loss of generality one can finally suppose that $t_2 > t_1$ which implies by Lemma 2.3.12 that $v_2 \geq v_1$ a.e. on B'. Moreover, $\mathcal{H}^{N-1}$-every $x' \in B'$ is a Lebesgue point of functions v_i, ∇v_i, $D^2 v_i$, $i \in \{1, 2\}$ thus at $\mathcal{H}^{N-1}$-almost every contact point one has

$$\begin{aligned} v_1(x') &= v_2(x'), \\ \nabla_{x'} v_1(x') &= \nabla_{x'} v_2(x'), \\ D^2_{x'} v_1(x') &\leq D^2_{x'} v_2(x'). \end{aligned} \tag{2.4.14}$$

Recall that Proposition 2.3.14 tells us that the superlevels E_{t_i} with $i \in \{1, 2\}$ solve

$$\min_E \int_{\partial^* E} w(x) d\mathcal{H}^{N-1}(x) + \int_E (t_i - g(x)) dx$$

where the minimization is carried out on all sets of finite perimeter in Ω. Doing compact modifications in the ball B' one immediately sees that v_i, $i \in \{1, 2\}$ minimizes

$$I(v) = \int_{B'} w(x', v_i(x')) \sqrt{1 + |\nabla_{x'} v_i(x')|^2} dx' + \int_{B'} \int_{v(x')}^{R} t_i - g(x', x_N) dx_N dx'. \tag{2.4.15}$$

This means that for any perturbation $\varphi \in C_c^\infty(B')$ such that $\varphi \geq 0$

$$I'(v_i)^+ \cdot \varphi = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{I(v_i + \varepsilon\varphi) - I(v_i)}{\varepsilon} \geq 0$$

whereas we know by the slicing properties of BV functions (see in particular [8, Remark 3.109]) that for a.e. $x' \in B'$

$$\frac{1}{\varepsilon}\int_{v_i(x')}^{v_i(x')+\varepsilon\varphi(x')} g(x', x_N)dx_N \underset{\varepsilon\to 0}{\to} g(x', v_i(x') + 0),$$
$$\frac{w(x', v_i(x') + \varepsilon\varphi(x')) - w(x', v_i(x'))}{\varepsilon} \underset{\varepsilon\to 0}{\to} \partial_{x_N} w(x', v_i(x') + 0).$$

Thus, we find that, for any $\varphi \in C_c^\infty(B')$ such that $\varphi \geq 0$,

$$\begin{aligned} I'(v_i)^+ \cdot \varphi &= \int_{B'} w(x', v_i(x'))\left(\frac{\nabla_{x'} v_i(x')}{\sqrt{1 + |\nabla_{x'} v_i(x')|^2}}\right) \cdot \nabla_{x'}\varphi(x')dx' \\ &+ \int_{B'} (\partial_{x_N} w(x', v_i(x') + 0)\sqrt{1 + |\nabla_{x'} v_i(x')|^2} - (t_i - g(x', v_i(x') + 0)))\varphi(x')dx'. \end{aligned} \tag{2.4.16}$$

Our aim is now to integrate by parts in the first integral that we shall denote $\tilde{I}(v_i)$. For this purpose, let us also denote $f_i(x') = w(x', v_i(x'))$. It is readily checked that $f_i \in \mathrm{Lip}(B') \subset H^1(B')$. Therefore, v_i being regular

$$\begin{aligned} \tilde{I}(v_i) &= \int_{B'} f_i\left(\frac{\nabla_{x'} v_i}{\sqrt{1 + |\nabla_{x'} v_i|^2}}\right)\nabla_{x'}\varphi \\ &= -\int_{B'} \mathrm{div}_{x'}\left(f_i\left(\frac{\nabla_{x'} v_i}{\sqrt{1 + |\nabla_{x'} v_i|^2}}\right)\right)\varphi \\ &= -\int_{B'} \nabla_{x'} f_i\left(\frac{\nabla_{x'} v_i}{\sqrt{1 + |\nabla_{x'} v_i|^2}}\right)\varphi - \int_{B'} w(\cdot, v_i)\kappa_i\varphi \end{aligned} \tag{2.4.17}$$

where we denoted $\kappa_i(x') = \mathrm{div}_{x'}\left(\frac{\nabla_{x'} v_i(x')}{\sqrt{1+|\nabla_{x'} v_i(x')|^2}}\right)$ the mean curvature of the level set $\partial^* E_{t_i}$ at $(x', v_i(x'))$.

Note that $\mathcal{H}^{N-1}$-a.e. point in B' is a Lebesgue point for $\nabla_{x'} f_i$ so v_i satisfies

$$\begin{aligned} &-\nabla_{x'} f_i(x') \cdot \frac{\nabla_{x'} v_i(x')}{\sqrt{1 + |\nabla_{x'} v_i(x')|^2}} - w(x', v_i(x'))\kappa_i(x') \\ &+\partial_{x_N} w(x', v_i(x') + 0)\sqrt{1 + |\nabla_{x'} v_i(x')|^2} - \big(t_i - g(x', v_i(x') + 0)\big) \geq 0. \end{aligned} \tag{2.4.18}$$

If one chooses $\varepsilon < 0$ in (2.4.15) then one obtains in the same way

$$\begin{aligned} &-\nabla_{x'} f_i(x') \cdot \frac{\nabla_{x'} v_i(x')}{\sqrt{1+|\nabla_{x'} v_i(x')|^2}} - w(x', v_i(x'))\kappa_i(x') \\ &+\partial_{x_N} w(x', v_i(x') - 0)\sqrt{1+|\nabla_{x'} v_i(x')|^2} - \big(t_i - g(x', v_i(x') - 0)\big) \leq 0. \end{aligned} \quad (2.4.19)$$

These identities hold for a.e. $x' \in B'$. Since we assumed that the contact set $\{v_1 = v_2\}$ has positive $\mathcal{H}^{N-1}$-measure, then we can find a contact point $x' \in B'$ that satisfies the inequalities (2.4.18) and (2.4.19), identities (2.4.14) and such that $\nabla_{x'} f_1(x') = \nabla_{x'} f_2(x')$ (indeed $f_1 = f_2$ on the contact set) which implies that

$$\begin{aligned} &\partial_{x_N} w(x', x_N - 0)\sqrt{1+|\nabla_{x'} v_1(x')|^2} - (t_1 - g(x', x_N - 0)) \leq w(x)\kappa_1(x') \\ &\leq w(x)\kappa_2(x') \leq \partial_{x_N} w(x', x_N + 0)\sqrt{1+|\nabla_{x'} v_2(x')|^2} - (t_2 - g(x', x_N + 0)). \end{aligned} \quad (2.4.20)$$

It follows that for $\mathcal{H}^{N-1}$-every $x' \in B'$

$$\begin{aligned} 0 < t_2 - t_1 \leq \big(\partial_{x_N} w(x', x_N + 0) - \partial_{x_N} w(x', x_N - 0)\big)(1+\eta) \\ +\big(g(x', x_N + 0) - g(x', x_N - 0)\big) \end{aligned}$$

with η that can be chosen as small as one wishes by taking a smaller ball B'. Thus $\partial_{x_N} w$ or g jumps at $\bar{x}$ hence (2.4.12). Moreover, from the previous inequality one has for the value of the jump

$$(u^+ - u^-)(x) \leq (\partial_{x_N} w^+ - \partial_{x_N} w^-)(x) + (g^+ - g^-)(x) \quad (2.4.21)$$

which furnishes (2.4.13). The claim on the mean curvature follows at once from (2.4.20). □

Remark 2.4.19.(i) Assume that the discontinuity of $\partial_{x_N} w$ occurs in the opposite direction of that of g, namely

$$(\partial_{x_N} w^+ - \partial_{x_N} w^-)(x) + (g^+ - g^-)(x) = 0.$$

Then one can simply erase the jump of g: indeed from (2.4.21) one sees that u has no discontinuity at x.

(ii) Note that if one is merely interested in the jump inclusion (2.4.12), it can be obtained by copying and pasting the proof given in the anisotropic setting: indeed reasoning by contradiction one can assume that $\mathcal{H}^{N-1}\big(J_u \setminus (J_g \cup J_{\nabla w})\big) > 0$ and the rest follows.

2.4.5 Dependency on the regularization parameter λ

In this section, we assume that the weight is constant *i.e.* $w = \lambda > 0$. This corresponds to ROF's model. Before getting further we need the following

Lemma 2.4.20. *Consider an open set Ω with finite Lebesgue measure, let $g \in L^\infty(\Omega)$ and consider respectively two minimizers u_λ, u_μ of (ROF) corresponding to the regularization parameters $\lambda,\ \mu > 0$ then*

$$\|u_\lambda - u_\mu\|_\infty \leq \frac{2|\Omega|\|g\|_\infty}{\min(\lambda, \mu)}|\lambda - \mu|.$$

Proof. Without loss of generality, one can assume that $\mu > \lambda$. The minimizers u_λ and u_μ satisfy the Euler-Lagrange equation for ROF *i.e.* there exist $z_\lambda, z_\mu \in L^\infty(\Omega, \mathbb{R}^N)$ such that

$$\begin{cases} -\lambda \operatorname{div} z_\lambda + u_\lambda = g, \\ -\mu \operatorname{div} z_\mu + u_\mu = g. \end{cases}$$

Multiplying the first equation by μ/λ, the second by -1 and summing the resulting identities we get

$$\langle -\mu \operatorname{div}(z_\lambda - z_\mu) + u_\lambda - u_\mu, \varphi \rangle = \left(\frac{\mu}{\lambda} - 1\right) \langle (g - u_\lambda), \varphi \rangle.$$

for any test function $\varphi \in L^2(\Omega)$. If for some *even* integer $p \geq 2$ we set $\varphi = (u_\lambda - u_\mu)^{p-1}$ and denote $q = p/(p-1)$ the adjoint of p, it follows

$$\begin{aligned} \mu\langle (z_\lambda - z_\mu), (p-1)(u_\lambda - u_\mu)^{p-2} D(u_\lambda - u_\mu)\rangle + \|u_\lambda - u_\mu\|_p^p \\ \leq \left(\frac{\mu}{\lambda} - 1\right) \|g - u_\lambda\|_q \|u_\lambda - u_\mu\|_p^{p-1}. \end{aligned}$$

Though, the first term on the left side of the inequality is non-negative since ∂TV is a monotone operator (see [41]) so we are simply left with

$$\|u_\lambda - u_\mu\|_p \leq \frac{\mu - \lambda}{\lambda} \|g - u_\lambda\|_q \tag{2.4.22}$$

which implies

$$\begin{aligned} \|u_\lambda - u_\mu\|_p &\leq |\Omega|^{\frac{1}{q}} \frac{\mu - \lambda}{\lambda} \|g - u_\lambda\|_\infty \\ &\leq 2|\Omega|^{\frac{1}{q}} \|g\|_\infty \frac{\mu - \lambda}{\lambda} \end{aligned} \tag{2.4.23}$$

which yields the result making $p \to +\infty$. ◻

Remark 2.4.21.(i) Equation (2.4.22) in conjunction with (2.2.1) implies that for $g \in BV(\Omega)$,

$$\|u_\lambda - u_\mu\|_2 \leq \frac{|\mu-\lambda|}{\min(\lambda,\mu)}\|g-u_\lambda\|_2 \leq \sqrt{2}\frac{\mu-\lambda}{\sqrt{\min(\lambda,\mu)}}\left(\int_\Omega |Dg|\right)^{\frac{1}{2}},$$

where the rightmost bound does not depend on $|\Omega|$ hence we can relax the assumption on Ω.

(ii) Inequality (2.4.23) suggests some differentiability property for the mapping

$$\begin{cases} \mathbb{R}^+ & \to L^p(\Omega) \\ \lambda & \mapsto u(\lambda) \end{cases}$$

defined for $p \in [2, +\infty]$. Unfortunately Rademacher's theorem fails in the infinite dimensional setting. Nonetheless, in our problem we can actually get Fréchet-differentiability almost everywhere from [25, Corollary 5.21] whenever the destination space has the so-called Radon-Nikodym Property (RNP). A space satisfies the RNP whenever it is a separable dual Banach space or a reflexive space (see [25, Corollary 5.12]). This is indeed true for any $L^p(\Omega)$ space with $p < \infty$ but not for $L^\infty(\Omega)$ and furnishes the differentiability for the $\|\cdot\|_p$ norm only.

Note that in general it is not trivial to get Fréchet-differentiability for a generic mapping with values in a space of infinite dimension. The only positive answer in this direction states that every real-valued Lipschitz function on an Asplund space has points of Fréchet differentiability (see [138] but also [29] for counterexamples). In general, the result does not even hold after convolution (see for instance [30]). Though the situation for Gâteaux is more favorable: the idea is that every Lipschitz map from a separable Banach space into a space with the RNP is Gâteaux differentiable almost everywhere in the sense of Aronszajn ([25, Proposition 6.41 and Theorem 6.42]).

Lemma 2.4.20 helps us prove the following result that says essentially that the highest jumps form a decreasing sequence with respect to the regularization parameter λ:

Proposition 2.4.22. *Let an open domain $\Omega \subset \mathbb{R}^N$ of finite Lebesgue measure, $g \in L^\infty(\Omega)$ non identically zero and λ, μ positive such that for some real $\varepsilon > 0$*

$$|\mu - \lambda| \leq \frac{\varepsilon \min(\lambda,\mu)}{2|\Omega|\|g\|_\infty}.$$

Let also u_λ and u_μ be two minimizers of (ROF). Then if we denote

$$J^\varepsilon_{u_\lambda} := \{x \in J_{u_\lambda} \ / \ (u^+_\lambda - u^-_\lambda)(x) > \varepsilon\},$$

one has

$$J^{\varepsilon}_{u_\lambda} \subset J_{u_\mu}$$

up to a $\mathcal{H}^{N-1}$*-negligible set.*

Proof. This proposition is a straightforward application of the preceding lemma which implies $\|u_\lambda - u_\mu\|_\infty \le \varepsilon$. Then clearly for $\mathcal{H}^{N-1}$-almost any $x \in J^{\varepsilon}_{u_\lambda}$

$$\varepsilon < (u^+_\lambda - u^-_\lambda)(x) \le \varepsilon + (u^+_\mu - u^-_\mu)(x)$$

hence the conclusion. $\square$

2.5 Staircasing for the denoising problem

From now on, we shall only focus on the (ROF) problem and drop the anisotropy. As recalled in our introduction, it has been long observed that the minimizer of this problem has unnatural homogeneous regions referred to as the *staircase* regions. The problem of proving the existence of these constant zones has been tackled in the discrete setting in [133] though almost nothing is known in the continuous setting. We will prove in this section that the staircasing phenomenon is unavoidable in the continuous setting in dimension $N \ge 2$ even though there is no addition of noise.

Staircasing through the level lines:

Figure 2.11: Level lines for the TV-minimizer of the *Lena* image

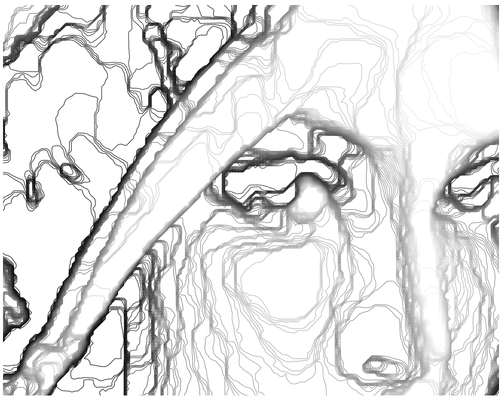


Figure 2.12: Level lines for the TV-minimizer of a noisy image

The previous images show that the level lines of the minimizers miss large regions that are therefore constant. The idea of looking at the level sets gives a good intuition but it is also a key point in our analysis to establish results from a theoretical point of view.

2.5.1 Staircasing at extrema

Let us start by stating one of the main results of this section:

Theorem 2.5.1. *Let $g \in L^2(\mathbb{R}^N)$ bounded from above. Then the associated minimizer u_λ of (ROF) ($\lambda > 0$) is also bounded from above, attains its maximum and one has*

$$|\{u_\lambda = \max u_\lambda\}| > 0.$$

In particular $Du_\lambda = 0$ in $\{u_\lambda = \max u_\lambda\}$.

Proof. We denote $m_g := \sup g$. Proving that there is a staircase amounts to show that the superlevel sets vanish at some point. Let us therefore consider the superlevel $E_t^\lambda := \{u_\lambda > t\}$ for some real t. By Theorem 2.2.13, we know that E_t^λ minimizes the following problem

$$\min_E \lambda P(E) + \int_E (t - g(x))dx.$$

In particular,

$$\lambda P\left(E_t^\lambda\right) + \int_{E_t^\lambda} (t - g(x))dx \leq 0. \tag{2.5.1}$$

By the isoperimetric inequality and equation (2.5.1), we get

$$|E_t^\lambda|^{\frac{N-1}{N}} \leq P\left(E_t^\lambda\right) \leq \frac{1}{\lambda}\int_{E_t^\lambda} (g(x) - t)dx \leq |E_t^\lambda|\frac{m_g - t}{\lambda}.$$

This implies immediately that $|E_t^\lambda| = 0$ for any $t \geq m_g$. Actually, by a thresholding argument, one can show that $E_t^\lambda = \emptyset$, but this additional information will not be needed here.

If for some $t \in \mathbb{R}$, $|E_t^\lambda| \neq 0$ then

$$|E_t^\lambda| \geq \left(\frac{\lambda}{m_g - t}\right)^N. \tag{2.5.2}$$

Let us assume that we have $|E_t^\lambda| \neq 0$ for any $t_0 < t < m_g$. Then letting $t \to m_g$ in (2.5.2) contradicts $|E_t^\lambda| < +\infty$. So, if we set $m_u := \sup u_\lambda$, we therefore have $m_u = \sup\{t, |E_t^\lambda| \neq 0\} < m_g$.

Now, we would like to prove that $|\{u_\lambda = m_u\}| \neq 0$. This is indeed true since

$$\{u_\lambda = m_u\} = \bigcap_{n \in \mathbb{N}} E^\lambda_{m_u - \frac{1}{n}}$$

and by (2.5.2) it follows

$$|\{u_\lambda = m_u\}| \geq \lim_{n\to\infty} \left|E^\lambda_{m_u - \frac{1}{n}}\right| \geq \left(\frac{\lambda}{m_g - m_u}\right)^N > 0.$$

By the coarea formula, we then simply have

$$\begin{aligned}\int_{\{u_\lambda = m_u\}} |Du_\lambda| &= \int_{-\infty}^{+\infty} P(E_t^\lambda \cap \{u_\lambda = m_u\})dt\\ &= \int_{m_u}^{m_g} P(E_t^\lambda)dt\\ &= 0.\end{aligned}$$

This implies that $Du_\lambda = 0$ on the staircase set $\{u_\lambda = m_u\}$ that has positive Lebesgue measure. □

Remark 2.5.2.(i) In case $g \in L^\infty(\Omega)$ is not assumed to be constant, we actually proved that u is also bounded and that we have a.e.

$$\inf_\Omega g < \min_\Omega u_\lambda \leq \max_\Omega u_\lambda < \sup_\Omega g.$$

Moreover, inequality (2.5.2) gives a lower bound for the staircasing effect namely

$$\begin{aligned}|\{Du_\lambda = 0\}| &\geq |\{u_\lambda = \min_\Omega u_\lambda\} \cup \{u_\lambda = \max_\Omega u_\lambda\}|\\ &\geq 2\left(\frac{\lambda}{\sup_\Omega g - \inf_\Omega g}\right)^N. \qquad (2.5.3)\end{aligned}$$

(ii) As was already stated in Theorem 2.2.13, it is possible to prove that $\{u_\lambda = \max_\Omega u_\lambda\}$ has an open representative which in turn implies $|\{u_\lambda = \max_\Omega u_\lambda\}| > 0$. Though the proof is not direct and relies on the density estimate established in Lemma 2.4.13 for quasi-minimizers of the perimeter. Needless to say that the same holds for $\{u_\lambda = \min_\Omega u_\lambda\}$.

As we just saw, Theorem 2.5.1 furnishes a way to quantify the staircase effect through the inequality (2.5.3). Nonetheless, this bound is not sharp as can be seen for g the characteristic of a convex set (see below). The reason is that we do not take into account creation of flat zones occurring near local extrema. This is the object of the following theorem

Theorem 2.5.3. *Let $g \in L^p(\Omega)$ with $p \in (N, +\infty]$ and u_λ with $\lambda > 0$ the corresponding minimizer of (ROF). If x_0 is a local extremum of u_λ then there exists $\mathcal{N}(x_0)$ a neighborhood of x_0 such that*

$$Du_\lambda = 0 \text{ on } \mathcal{N}(x_0).$$

Proof. Without loss of generality, we can consider a local maximum point x_0 of u_λ that is to say there is a radius $\rho > 0$ such that

$$u_\lambda \leq u_\lambda^+(x_0) \text{ on } B(x_0, \rho).$$

Considering that $x_0 \in E_t^\lambda = \{u_\lambda > t\}$ for some $t < u_\lambda^+(x_0)$, one has for any $r > 0$,

$$|E_t^\lambda \cap B(x_0, r)| > 0,$$

and by the density lemma there exists $r_0 > 0$ such that

$$|E_t^\lambda \cap B(x_0, r)| \geq \frac{w_N r^N}{2^N},$$

for any $0 < r \leq r_0$. Now, if we take $r = \min(\rho, r_0)$ and make $t \to u_\lambda^+(x_0)$, we get

$$|\{u_\lambda = u_\lambda^+(x_0)\} \cap B(x_0, \rho)| \geq \frac{w_N r^N}{2^N}.$$

Moreover, according to Remark 2.5.2, $\{u_\lambda = u_\lambda^+(x_0)\}$ has an open representative. $\square$

Putting together these two theorems we get

Corollary 2.5.4. *Let $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and u_λ, $\lambda > 0$ be the minimizer of (ROF) associated to g. If u_λ is constant on at most two disjoint sets then it has no local extrema other than its global maximum and minimum.*

2.5.2 Dependency of the staircasing on λ

In the previous section, we proved that for fixed λ staircasing always occurs and can be quantified by (2.5.1). This bound suggests that the Lebesgue measure of the created flat zones is non-decreasing with respect to λ. This was already observed for the total variation flow in $\mathbb{R}^N$. In [13, Chapter 4] the authors even prove that the solution $u(t)$ of the total variation flow in $\mathbb{R}^N$ decreases in time, for some norm, with a finite extinction time. It is possible to get a similar result for the minimizer u_λ:

Proposition 2.5.5. *Let Ω be a connected bounded Lipschitz continuous open set in $\mathbb{R}^N$, $g \in L^N(\Omega)$ and u_λ the minimizer of (ROF). Then, there exists $\lambda^* = C_\Omega \|g\|_N \geq 0$ with C_Ω that only depends on Ω, such that for any $\lambda \geq \lambda^*$,*

$$u_\lambda = \frac{1}{|\Omega|} \int_\Omega g.$$

Proof. The conclusion follows readily if one can find some z_λ that satisfies the system

$$\begin{cases} -\lambda \operatorname{div} z_\lambda + \frac{1}{|\Omega|}\int_\Omega g = g & \text{in } \Omega, \\ \|z_\lambda\|_\infty \le 1 & \text{in } \Omega, \\ z_\lambda \cdot \nu_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

If $p \ge N$, the function $g - \frac{1}{|\Omega|}\int_\Omega g$ is in $L^N(\Omega)$ and of mean zero. Therefore, the result of Bourgain and Brezis [37] (see also [72]) asserts that there exists a $z \in L^\infty(\Omega, \mathbb{R}^N) \cap W^{1,N}(\Omega, \mathbb{R}^N)$ that solves

$$\begin{cases} -\operatorname{div} z = g - \frac{1}{|\Omega|}\int_\Omega g & \text{in } \Omega, \\ z \cdot \nu_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus a nice candidate is $z_\lambda = \frac{z}{\lambda}$, for λ large enough. Actually, $\lambda^* = \|z\|_\infty$ that is controlled by $\|g\|_N$.

□

If the domain is not bounded, the result of Bourgain and Brezis does not apply. Though, one has a similar result for a data g that lies in the so-called *Schwartz class* $\mathcal{S}$ that contains those functions whose derivatives are decreasing faster that any polynomial (see [97]):

Proposition 2.5.6. *If g is an element of $\mathcal{S}(\mathbb{R}^N)$ and u_λ denotes the minimizer of (ROF) corresponding to g then there exists $\lambda^* \ge 0$ such that for $\lambda \ge \lambda^*$*

$$u_\lambda = 0 \textit{ in } \mathbb{R}^N.$$

Proof. As in the previous proof, the assertion follows if one can find some z_λ that solves the following system

$$\begin{cases} -\lambda \operatorname{div} z_\lambda = g, \\ \|z_\lambda\|_\infty \le 1. \end{cases}$$

Let us look for a z_λ that is of the form $z_\lambda = \frac{\nabla v}{\lambda}$ with v that satisfies in $\mathbb{R}^N$

$$-\Delta v = g. \tag{2.5.4}$$

Taking the Fourier transform on both sides, one gets

$$-4\pi^2 |\xi|^2 \hat{v}(\xi) = \hat{g}(\xi), \ \forall \xi \in \mathbb{R}^N$$

hence the following estimate

$$\|\nabla v\|_\infty \leq \|\widehat{\nabla v}\|_1 = \frac{1}{4\pi^2}\left\||\xi|^{-1}\hat{g}(\xi)\right\|_1 < +\infty$$

where in the last inequality we used the well-known fact that $\hat{g} \in \mathcal{S} \subset L^p(\mathbb{R}^N)$ for any $p \in [1,\infty]$ (see [97] for instance for further details). Therefore $z_\lambda = \frac{\nabla v}{\lambda}$ satisfies the system above as soon as $\lambda \geq \lambda^* := \|\nabla v\|_\infty$. $\square$

For a general unbounded subdomain of $\mathbb{R}^N$, the previous proof cannot be adapted since it is well known that for a bounded g, equation (2.5.4) does not necessarily admit a solution v in $W^{2,\infty}$ (see [37]).

In case $N = 2$, it is possible to prove Proposition 2.5.5 without having to use the difficult result of Bourgain and Brezis:

Proposition 2.5.7. *Let $\Omega \subset \mathbb{R}^2$ a be connected open set that is bounded with a Lipschitz continuous boundary, $g \in L^2(\Omega)$ and u_λ the minimizer of (ROF) associated to g. Then there exists $\lambda^* = C_\Omega\|g\|_2$, with C_Ω that only depends on Ω, such that for $\lambda \geq \lambda^*$*

$$u_\lambda = \frac{1}{|\Omega|}\int_\Omega g.$$

Proof. Given $u \in BV(\Omega) \cap L^2(\Omega)$, in [13, Lemma 2.4], one can find the following characterization of $p = -\operatorname{div}(z) \in \partial TV(u)$:

$$\int_\Omega |Du| \leq \int_\Omega (u-\varphi)p + \int_\Omega z\cdot\nabla\varphi,\ \forall\varphi \in W^{1,1}(\Omega)\cap L^2(\Omega).$$

Therefore, denoting $\langle u_\lambda\rangle$ the average of u_λ and setting $u = u_\lambda$, $\varphi = \langle u_\lambda\rangle$ and $p = \frac{1}{\lambda}(g - u_\lambda)$ one has

$$\int_\Omega |Du_\lambda| \leq \frac{1}{\lambda}\int_\Omega (g-u_\lambda)\left(u_\lambda - \langle u_\lambda\rangle\right).$$

Then, applying the Poincaré inequality and Cauchy-Schwarz and using $\langle u_\lambda\rangle = \langle g\rangle$, the average of g, one obtains the estimate

$$C\|u_\lambda - \langle g\rangle\|_2 \leq \frac{1}{\lambda}\|u_\lambda - \langle g\rangle\|_2\|u_\lambda - g\|_2,$$

where C is the constant that appears in the Poincaré inequality. Now remarking that, by minimality of u_λ,

$$\|u_\lambda - g\|_2 \leq \|g\|_2$$

concludes the proof. Moreover, we get $\lambda^* = \frac{\|g\|_2}{C}$. $\square$

Remark 2.5.8. This way we also get the *calibration* z_λ through the Euler-Lagrange equation.

Since there exists a version of the Poincaré inequality in $\mathbb{R}^N$, for which the optimal constant is known, it is readily checked that we also gave an alternative proof for Proposition 2.5.6 in case $N = 2$:

Proposition 2.5.9. *Let $g \in L^2(\mathbb{R}^2)$ and u_λ be the minimizer of (ROF) associated to g then there exists*

$$\lambda^* = \left(2\pi^{\frac{1}{2}}\right)^{-1} \|g\|_2$$

such that for $\lambda \geq \lambda^*$

$$u_\lambda = 0 \text{ in } \mathbb{R}^N.$$

Remark 2.5.10.(i) Reasoning as we did we get an explicit λ^* that is optimal as can be seen for $g = \chi_{\mathbb{D}}$ the characteristic of the unit disc since we know in this case that

$$u_\lambda = (1 - 2\lambda)^+ \chi_{\mathbb{D}}.$$

(ii) Our proof is very similar in spirit to that one given in [13, Theorem 2.21] for the total variation flow when again $N = 2$. It was brought to our attention that in a recent article (see [88]), Giga and Kohn extend the results of [13] in the N-dimensional case. We thus suspect that their proof can be adapted to our context.

Now that we got rid of the case when λ is large, let us see through some examples how the staircase regions behave for reasonable values of the regularization parameter. In the rest of this section, we assume that g is the characteristic function of a set C which means we are now interested in the minimizers of

$$\mathcal{E}_\lambda(u) = \lambda \int_\Omega |Du| + \frac{1}{2}\|u - \chi_C\|_2^2.$$

We are going to distinguish two different cases for C:

The characteristic of a bounded convex set in $\mathbb{R}^N$.

Then it is well known (see [2, 5]) that for $\lambda \leq \lambda^*$ and for any $t \geq 0$ the superlevel $E_t^\lambda = \{u_\lambda > t\}$ of the solution u_λ is given by

$$E_t^\lambda = \begin{cases} C_{\lambda/(1-t)} \text{ if } t \leq 1 - \lambda/R^*, \\ \emptyset \text{ otherwise.} \end{cases}$$

where for any $R > 0$, C_R is the opening of C defined by

$$C_R = \bigcup_{B(x,R)\subset C} B(x, R),$$

and R^* is the inverse of the so-called *Cheeger constant* defined by the value of R that solves

$$\frac{P(C_R)}{|C_R|} = \frac{1}{R}.$$

Therefore, the staircase set

$$\{u_\lambda = \max u_\lambda\} = C_{R^*}$$

is the so-called *Cheeger set* and is independent of $\lambda \leq \lambda^*$. We would get similar results if C were the union of spaced convex sets (see [13] for the expression of u_λ in this case).

The characteristic of two touching squares in $\mathbb{R}^2$.

Here $C = [0, 1] \times [0, -1] \cup [-1, 0] \times [0, 1]$ is the union of two unit squares that only touch on a vertex. In [3], Allard gives a full description of the superlevels of the solution u_λ, hence u_λ itself. The level sets $E_t^\lambda = \{u_\lambda > t\}$ are of five kinds namely

$$E_t^\lambda \in \left\{\emptyset,\ \mathbb{R}^2,\ F_{r,s},\ G_{r,s},\ H_r\ /\ \lambda = \frac{1}{r} + \frac{1}{s}, r, s \in \mathbb{R}^+\right\}$$

where the last three sets are formally defined in [3]. They are depicted in the following figures as the interior of the domain bounded by the red curve:

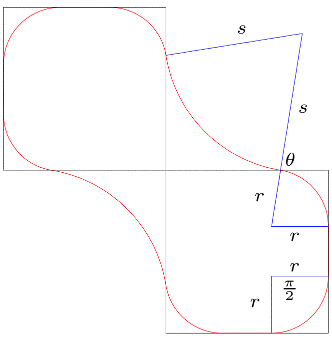


Figure 2.13: $F_{r,s}$.

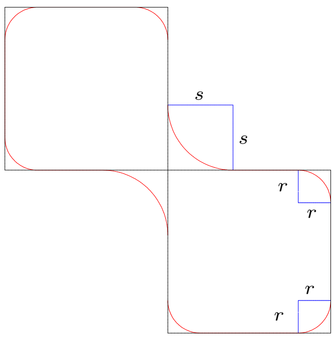


Figure 2.14: $G_{r,s}$.

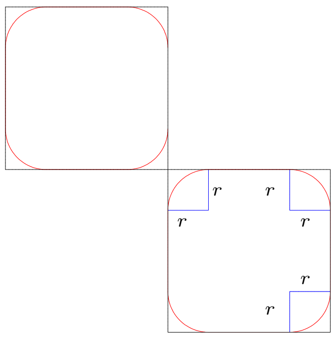


Figure 2.15: H_r.

In the following figure Allard summed up the different possibilities:

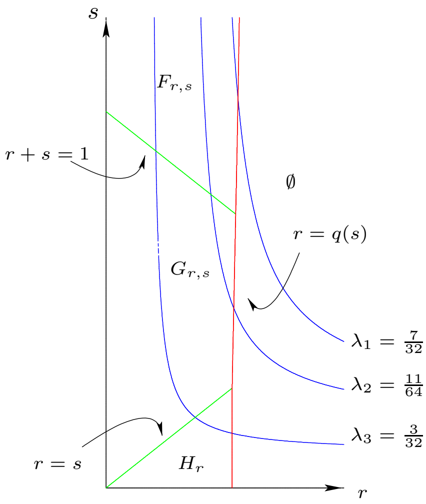


Figure 2.16: Level sets E_t^λ.

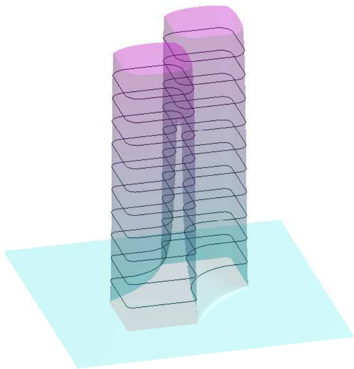


Figure 2.17: u_λ and its level lines.

Now consider that $\lambda \in (0, \lambda^*)$. As seen previously, the staircase set $S_\lambda := \{u_\lambda = \max u_\lambda\}$ is exactly the smallest superlevel set that is not empty this is to say

$$S_\lambda \in \{F_{q(s),s},\ G_{q(s),s},\ H_{q(s)}\}$$

where the function $q(s)$ is non-decreasing in s. Let us focus on two values of the regularization parameter: $\lambda_1 = \frac{7}{32}$ and $\lambda_2 = \frac{11}{64}$ and let $s_i,\ i \in \{1,2\}$ be the unique value such that $\lambda_i = \frac{1}{s_i} + \frac{1}{q(s_i)}$. Then it is readily seen that

$$S_{\lambda_2} = G_{q(s_2),s_2} \not\subset F_{q(s_1),s_1} = S_{\lambda_1}$$

even though $\lambda_2 < \lambda_1$.

This example shows that in general the staircase zones $(S_\lambda)_{\lambda \geq 0}$ do not form a monotone sequence.

Such a phenomenon cannot occur given a radial function g and this is what we are going to prove in the following section. In short, this is due to the fact that the solutions $(u_\lambda)_\lambda$ of the radial problem form a semi-group. This was already established in the one-dimensional setting [43] and in case $g = \chi_C$ the characteristic function of a convex set C [13, 5].

2.6 Denoising problem for radial data

Unless otherwise stated, in this section Ω will denote the ball $B(0,R) \subset \mathbb{R}^N$ with $N \geq 2$ and we consider a radial $g \in L^2(\Omega)$. It is easily seen that the minimizer u_λ of (ROF) is itself radial. Indeed one could argue that for any rotation $\mathcal{R}$,

$u_\lambda(\mathcal{R}x)$ is also a minimizer. Since it is unique $u_\lambda(\mathcal{R}x) = u_\lambda(x)$ for any $x \in \mathbb{R}^N$. We denote $\tilde{v}$ the function defined by $\tilde{v}(|x|) = v(x)$ for any $x \in \mathbb{R}^N$. Then u_λ minimizes

$$\min_{u \in BV(\Omega)} \int_0^R \left(\lambda |\tilde{u}'(r)| + \frac{1}{2}(\tilde{u}(r) - \tilde{g}(r))^2 \right) r^{N-1} dr. \tag{2.6.1}$$

Thus, proceeding as in the one-dimensional case, either u_λ is constant or

$$u_\lambda(x) = g(x) + \operatorname{sgn}\left(\tilde{u}_\lambda'\big(|x|\big)\right) \frac{(N-1)\lambda}{|x|}$$

for any $x \in \mathbb{R}^N \setminus \{0\}$. Now, introducing the dual variable $\tilde{z}$ and reasoning as in [54], we can derive a dual formulation for the minimization problem (2.6.1), that is

$$\inf_{\substack{\tilde{z} \in W^{1,2}(0,R), \\ \tilde{z}(R)=0, \ |\tilde{z}| \le \lambda}} \int_0^R \left(\tilde{z}'(r) + \frac{N-1}{r} \tilde{z}(r) + g(r) \right)^2 r^{N-1} dr.$$

Then, if we set $z(x) = \tilde{z}(x) \frac{x}{|x|}$ for any $x \in \mathbb{R}^N \setminus \{0\}$,

$$(\operatorname{div}(z))(x) = \nabla(\tilde{z}(|x|)) \cdot \frac{x}{|x|} + \tilde{z}(|x|) \operatorname{div}\left(\frac{x}{|x|}\right) = \tilde{z}'(|x|) + \tilde{z}(|x|) \frac{N-1}{|x|}$$

which gives the dual formulation

$$\inf_{z \in A_\lambda} \mathcal{E}(z) = \int_{B(0,R)} (\operatorname{div}(z) + g)^2, \tag{2.6.2}$$

where

$$A_\lambda = \left\{ z(x) = \tilde{z}(x) \frac{x}{|x|} \ / \ \forall x \in \mathbb{R}^N \setminus \{0\}, \tilde{z} \in W^{1,2}(0,R), \ \tilde{z}(R) = 0, \ |\tilde{z}| \le \lambda \right\}$$

is the set of admissible z. Henceforth, we denote z_λ a minimizer of problem (2.6.2). Since it is radial, we will sometimes write z_λ instead of $\tilde{z}_\lambda$. Note also that z_λ is actually continuous by the Sobolev embedding therem.

Remark 2.6.1. We decided to proceed this way to justify rigorously that one can pick a radial vectorfield z_λ in the Euler-Lagrange equation (2.2.2).

2.6.1 The solutions form a semi-group

Let us start with the following lemma:

Lemma 2.6.2. *Let $g \in L^p(\Omega)$ with $p \in [N, +\infty]$ then for any $\lambda > 0$,*

$$|z_\lambda(x)| \leq C|x|^{1-\frac{N}{p}} \|g - u_\lambda\|_{L^p(B(0,|x|))}$$

for some positive C hence in particular $z_\lambda(0) = 0$.

Proof. For a.e. $r = |x|$ with $x \in \mathbb{R}^N$

$$\operatorname{div}(z_\lambda)(x) = \tilde{z}_\lambda'(r) + \frac{N-1}{r}\tilde{z}_\lambda(r) = \frac{1}{r^{N-1}}(\tilde{z}_\lambda(r) r^{N-1})'.$$

Hence integrating with respect to r and using $\operatorname{div}(z_\lambda) \leq |g - u_\lambda|$, it follows

$$\tilde{z}_\lambda(r) r^{N-1} \leq \int_{B(0,r)} |g - u_\lambda| \leq C\|g - u_\lambda\|_{L^p(B(0,r))} r^{N\left(1-\frac{1}{p}\right)}$$

for some positive real C. $\square$

The following proposition is the key result in the study of the radial problem:

Proposition 2.6.3 (Comparison result)**.** *Let $g \in L^2(\Omega)$, $\mu > \lambda \geq 0$ and consider respectively z_λ and z_μ the corresponding minimizers of $\mathcal{E}$ then one has*

$$z_\lambda - \lambda \geq z_\mu - \mu.$$

Proof. On the one hand,

$$\begin{aligned}
\mathcal{E}\left(\left(z_\mu + (\lambda - \mu)\frac{x}{|x|}\right) \vee z_\lambda\right) &= \int_{\{z_\lambda > z_\mu + \lambda - \mu\}} (\operatorname{div}(z_\lambda) + g)^2 dx \\
&+ \int_{\{z_\lambda < z_\mu + \lambda - \mu\}} \left(\operatorname{div}\left(z_\mu + (\lambda - \mu)\frac{x}{|x|}\right) + g\right)^2 dx, \\
\mathcal{E}\left(\left(z_\lambda + (\mu - \lambda)\frac{x}{|x|}\right) \wedge z_\mu\right) &= \int_{\{z_\lambda > z_\mu + \lambda - \mu\}} (\operatorname{div}(z_\mu) + g)^2 dx \\
&+ \int_{\{z_\lambda < z_\mu + \lambda - \mu\}} \left(\operatorname{div}\left(z_\lambda + (\mu - \lambda)\frac{x}{|x|}\right) + g\right)^2 dx,
\end{aligned}$$

thus by minimality of z_μ and z_λ

$$\begin{aligned}
&\int_{\{z_\lambda < z_\mu + \lambda - \mu\}} \left((\operatorname{div}(z_\lambda) + g)^2 + (\operatorname{div}(z_\mu) + g)^2\right) dx \\
&\leq \int_{\{z_\lambda < z_\mu + \lambda - \mu\}} \left(\operatorname{div}\left(z_\mu + (\lambda - \mu)\frac{x}{|x|}\right) + g\right)^2 dx \\
&+ \int_{\{z_\lambda < z_\mu + \lambda - \mu\}} \left(\operatorname{div}\left(z_\lambda + (\mu - \lambda)\frac{x}{|x|}\right) + g\right)^2 dx,
\end{aligned}$$

and then it follows that

$$\int_{\{z_\lambda - z_\mu + \mu - \lambda < 0\}} (\mu - \lambda) \frac{N-1}{|x|} \left(\operatorname{div} \left(z_\lambda - z_\mu + (\mu - \lambda) \frac{x}{|x|} \right) \right) dx \geq 0. \qquad (2.6.3)$$

But on the other hand, given that $z_\lambda - z_\mu + \mu - \lambda > 0$ on $\partial\Omega$ and $z_\lambda - z_\mu + \mu - \lambda = 0$ on $\partial\{z_\lambda - z_\mu + \mu - \lambda < 0\}$, one gets by integration for $\varepsilon > 0$,

$$\begin{aligned} &\int_{[\Omega \setminus B(0,\varepsilon)] \cap \{z_\lambda - z_\mu + \mu - \lambda < 0\}} \frac{N-1}{|x|} \left(\operatorname{div} \left(z_\lambda - z_\mu + (\mu - \lambda) \frac{x}{|x|} \right) \right) dx \\ &\quad = -\frac{N-1}{\varepsilon} \int_{\partial B(0,\varepsilon) \cap \{z_\lambda - z_\mu + \mu - \lambda < 0\}} (z_\lambda - z_\mu + \mu - \lambda)(x) dx \\ &\quad + \int_{[\Omega \setminus B(0,\varepsilon)] \cap \{z_\lambda - z_\mu + \mu - \lambda < 0\}} \frac{N-1}{|x|^2} (z_\lambda - z_\mu + \mu - \lambda)(x) dx. \end{aligned}$$

Now one has to distinguish two cases:

If $N \geq 3$, then, given that $C := \|z_\lambda - z_\mu + \mu - \lambda\|_\infty < +\infty$, one has

$$0 \leq -\frac{N-1}{\varepsilon} \int_{\partial B(0,\varepsilon) \cap \{z_\lambda - z_\mu + \mu - \lambda < 0\}} (z_\lambda - z_\mu + \mu - \lambda)(x) dx \leq 2\pi C \varepsilon^{N-2}.$$

Thus sending $\varepsilon \to 0$ and considering identity (2.6.3) we obtain

$$\int_{\{z_\lambda - z_\mu + \mu - \lambda < 0\}} \frac{N-1}{|x|^2} (z_\lambda - z_\mu + \mu - \lambda)(x) dx \geq 0$$

hence

$$z_\lambda - \lambda \geq z_\mu - \mu.$$

If $N = 2$ and ε is small then, by Lemma 2.6.2,

$$\partial B(0,\varepsilon) \cap \{z_\lambda - z_\mu + \mu - \lambda < 0\} = \emptyset$$

thus

$$-\frac{N-1}{\varepsilon} \int_{\partial B(0,\varepsilon) \cap \{z_\lambda - z_\mu + \mu - \lambda < 0\}} (z_\lambda - z_\mu + \mu - \lambda)(x) dx = 0$$

and the conclusion follows in the same way as above. □

Corollary 2.6.4. *For any $\lambda, \mu \geq 0$, one has*

$$\|z_\lambda - z_\mu\|_\infty \leq |\lambda - \mu|.$$

Another very important consequence of Proposition 2.6.3 is that solutions

$(u_\lambda)_{\lambda\in\mathbb{R}^+}$ form a semigroup. To prove this, let us denote

$$T_\lambda(g) = \underset{u\in BV(\Omega)}{\operatorname{argmin}}\ \lambda \int_\Omega |Du| + \frac{1}{2}\|u-g\|_2^2$$

for any real $\lambda \geq 0$ and any datum $g \in L^2(\Omega)$. Thus,

$$T_\lambda = (I + \lambda\partial TV)^{-1}$$

is the *resolvent operator* discussed in [41].

Proposition 2.6.5. *If $\Omega = B(0,R)$, $g \in L^2(\Omega)$ is radial, $\mu > \lambda \geq 0$ are two regularization parameters then*

$$T_\mu(g) = T_{\mu-\lambda} \circ T_\lambda(g).$$

Proof. Step 1: Let us set

$$z_0 = \underset{z\in A_{\mu-\lambda}}{\operatorname{argmin}} \int_\Omega (\operatorname{div}(z) + T_\lambda(g))^2$$

and $z'_\mu = z_0 + z_\lambda$. We claim that $\operatorname{div}(z'_\mu) = \operatorname{div}(z_\mu)$.
On the one hand, we know from the previous corollary that $z_\mu - z_\lambda \in A_{\mu-\lambda}$ thus by comparison with z_0

$$\int_\Omega (\operatorname{div}(z'_\mu - z_\lambda) + T_\lambda(g))^2 \leq \int_\Omega (\operatorname{div}(z_\mu - z_\lambda) + T_\lambda(g))^2.$$

and using the Euler-Lagrange equation it follows that

$$\int_\Omega (\operatorname{div}(z'_\mu) + g)^2 \leq \int_\Omega (\operatorname{div}(z_\mu) + g)^2.$$

Whereas, on the other hand, $z'_\mu \in A_\mu$ thus by comparison with z_μ

$$\int_\Omega (\operatorname{div}(z_\mu) + g)^2 \leq \int_\Omega (\operatorname{div}(z'_\mu) + g)^2$$

which proves our claim.
Step 2: Now considering the following Euler-Lagrange equations

$$\begin{cases} \operatorname{div}(z_\mu) = T_\mu(g) - g, \\ \operatorname{div}(z_\lambda) = T_\lambda(g) - g, \\ \operatorname{div}(z_0) = T_{\mu-\lambda} \circ T_\lambda(g) - T_\lambda(g), \end{cases}$$

the conclusion follows readily from the result of Step 1. $\square$

Let us recall that given an open set $\Omega \subset \mathbb{R}^N$ and an initial condition $u(0) = g \in L^2(\Omega)$ there exists a unique solution to the gradient flow equation (see [13]):

$$-\partial_t u(t) \in \partial TV(u(t)) \text{ a.e. in } t \in [0,T]. \tag{2.6.4}$$

We are also going to need the following classical result for the resolvent operator of a maximal monotone operator (see [41, Corollary 4.4]):

Proposition 2.6.6. *Let Ω be an open set of $\mathbb{R}^N$ and $u(t)$ be the solution of (2.6.4) with an initial condition $u(0) = g \in L^2(\Omega)$ then*

$$\lim_{n\to+\infty} T_{\frac{t}{n}}{}^n(u(0)) = \lim_{n\to+\infty} \left(I + \frac{t}{n}\partial TV\right)^{-n} \big(u(0)\big) = u(t),$$

the convergence taking place in $L^2(\Omega)$.

With these results in hands we are ready to prove the main result of this section, namely

Theorem 2.6.7. *Let $\Omega = B(0,R)$, $g \in L^2(\Omega)$ be radial and $u(t)$ be the solution of (2.6.4) with an initial condition $u(0) = g$. Then $u(t)$ is the unique minimizer of*

$$\min_{u\in BV(\Omega)} t\int_\Omega |Du| + \frac{1}{2}\|u-g\|_2^2. \tag{2.6.5}$$

Proof. From Proposition 2.6.5 it follows

$$T_{\frac{t}{n}}{}^n(u(0)) = T_t(u(0))$$

and making $n \to +\infty$ one has by Proposition 2.6.6

$$u(t) = T_t(u(0))$$

hence the statement of the theorem. $\square$

Remark 2.6.8. This property is not true for general semi-groups $u(t)$. Let us reason by contradiction. If $u(t)$ solves both (2.6.4) and (2.6.5) then writing the Euler-Lagrange equation one has for some fixed $t > 0$:

$$\begin{cases} -\partial_t u(t) \in \partial TV(u(t)), \\ \frac{g-u(t)}{t} \in \partial TV(u(t)). \end{cases}$$

Placing ourselves on the set $\{\nabla u(t) \neq 0\}$, it follows

$$-t\partial_t u = g - u,$$

hence

$$\frac{d}{dt}\left(\frac{u}{t}\right) = -\frac{1}{t^2}(u - t\partial_t u) = -\frac{g}{t^2} = \frac{d}{dt}\left(\frac{g}{t}\right).$$

Thus, there exists $C(x)$ that does not depend on t such that

$$u(t,x) = g(x) + C(x)t$$

whenever $\nabla u(t) \neq 0$ in a neighborhood of x. This contradicts the example of Allard where $\Omega = \mathbb{R}^2$ and $g = \chi_{[0,1]\times[0,-1]\cup[-1,0]\times[0,1]}$. Indeed let us consider the origin $x_0 = 0$. Then from the analysis of Allard (see section 2.5.2) one knows that for $0 < t < t^*$ there is a neighborhood of x_0 that is contained in $\{\nabla u(t) \neq 0\}$. Thus we get a contradiction if the minimizer u_t of (ROF) with parameter $\lambda = t$ is not affine in t at x_0. This is indeed observed by a simple numerical experimentation:

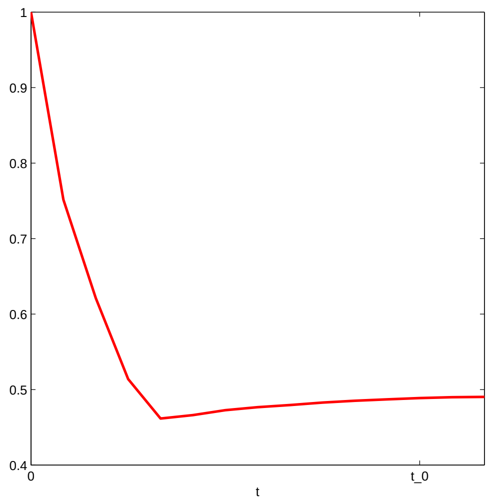


Figure 2.18: Evolution of $u(t,0)$

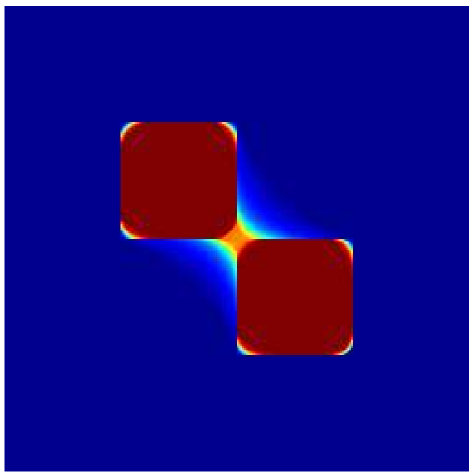


Figure 2.19: Solution $u(t)$ for $t = t_0$

The approach is rigorous since the algorithm used for the minimization of discrete (ROF) functional is convergent (see Chapter 5) and in [155] it is shown that the difference between the continuous (ROF) model and its finite difference discretization is bounded and tends to zero.

2.6.2 Staircasing and discontinuities

From Proposition 2.6.3 we can also get some information on the staircase regions. Indeed, whenever condition $|z_\lambda| \leq \lambda$ is saturated, we know from the Euler-Lagrange equation that $\nabla u_\lambda \neq 0$. This way, we can get an inclusion principle for the staircasing namely:

Theorem 2.6.9. *For any reals $\mu > \lambda \geq 0$, one has $\{|z_\lambda| < \lambda\} \subset \{|z_\mu| < \mu\}$.*

This is not true in general as seen through the example of Allard (see Section 2.5.2).

Remark 2.6.10. Theorem 2.6.7 is a key step in [43] to prove that in the one-dimensional case, staircasing occurs almost everywhere for a noisy 1D signal. By noisy we mean that the original signal was perturbed by a Wiener process. We cannot expect to get from our analysis such a result simply because a radial function that underwent an addition of noise is not radial anymore. We could though extend their result to the case of a radial "noise" but this does not seem really interesting. However, it would tell us again that staircasing is an important phenomenon for minimizers of perturbed signals.

As for the discontinuities one can actually refine the results of Section 2.4 thanks to the following

Theorem 2.6.11. *Let $\Omega = B(0,R)$ and $g \in L^N(\Omega)$ be radial. Consider also $\mu > \lambda > 0$ and u_λ, u_μ the corresponding minimizers of (ROF). Then one has*

$$J_{u_\mu} \subset J_{u_\lambda}.$$

If in addition, $g \in L^N(\Omega) \cap BV(\Omega)$ then given $\mu > \lambda \geq 0$,

$$J_{u_\mu} \subset J_{u_\lambda} \subset J_g.$$

Proof. Once we know Theorem 2.6.7 the result is a straightforward consequence of [48, Theorem 4.1] and [50, Theorem 2] which establishes a similar inclusion principle for the TV flow. □

2.7 Conclusion and perspective

In this chapter, we examined some fine results for energies involving terms that behave like the total variation. In particular, we prove that no new discontinuities are created for energies involving a smooth elliptic anisotropy and a generic fidelity term. This extends the result of [48] where they dealt with the denoising problem. On the other hand, we characterized creation of unobserved discontinuities for the adaptive total variation functional if the weight is merely Lipschitz continuous. In addition, we proved that the infinity norm is decreased at the discontinuity while minimizing ROF's energy, which is quite counterintuitive.

In the second part, we established that the staircasing phenomenon always occurs for the continuous ROF problem. We refined this result in the radial case by proving that the staircase zones are non-decreasing with the regularization parameter. The argument is based on the relation between the ROF problem and the total variation flow, which we prove to hold for radial data. In particular, using known results for the flow we also get that discontinuities form a monotone sequence given that the datum is radial.

The aforementioned results motivate many interesting questions that remain unsettled and pave the way for future researches. First of all, most of the results of this chapter (jump inclusion and staircasing) rely heavily on the connection with the perimeter problem via the coarea formula and it does not seem clear to us how they can be adapted to take into account linear perturbations of the data (convolution but also Radon or Fourier transforms).

Concerning the problem of inclusion of the discontinuities, it is not clear whether the discontinuities form a monotone sequence for a general datum. Indeed, in this case the connection with the flow fails. This question seems to be related to the existence of a smooth underlying calibration z (obviously not C^1) for the ROF problem. This question is actually interesting by itself. But "Finding a calibration remains an art, not a science" as would say Frank Morgan. Here, we should also mention the work of Bourgain-Brezis [37] and De Pauw-Pfeffer [72] where the authors were interested in finding a continuous z such that

$$\mathrm{div}(z) = \mu$$

for a given Radon measure μ. Though these results are not constructive since referring to the axiom of choice and cannot be easily adapted. The inclusion could also be obtained by establishing strong properties of the derivative $u'(\lambda)$ by means of Γ-convergence for instance (see [38]). Though the resulting functional seems to be non-local making the problem difficult (see [46]).

As for the staircasing phenomenon, it would be interesting to prove an N-dimensional counterpart for the result of the recent paper [43]: staircasing occurs almost everywhere for noisy images.

Another problem, which seems within reach, would be to examine the regularity of the minimizer for the general energy we considered. This could be done by adapting [49] where such a result is established for the denoising problem.

CHAPTER 3

An Alternative for the Total Variation with Applications in Imaging

Contents

3.1 Introduction

Functions of bounded variation are of particular interest in image processing. Ideally, a grayscale image can be seen as a function of the unit square into the the interval $[0, 1]$ representing the gray levels. Moreover, one wishes this function to be mostly smooth but one allows discontinuities all along regular curves: the edges of the objects that are depicted in the image. The derivative of such a function will then have measure parts, as functions of bounded variation. Formally, an integrable function u defined on an open set $\Omega \subset \mathbb{R}^N$ is of bounded variation on Ω, and we shall denote $u \in BV(\Omega)$, if the total variation

$$\int_\Omega |Du| < +\infty$$

The derivative Du being obviously taken in the sense of distributions.

Then, given a function $u \in BV(\Omega)$, it is natural to be interested in the quantity

$$J(u) = \inf_{P\phi = Du} \int_\Omega |\phi|$$

where the operator P is the "orthogonal projection on gradients" and acts on measures ϕ. To better understand this projection, let us for the moment consider that ϕ is simply a function in $L^2(\Omega, \mathbb{R}^N)$ and one assumes in addition that the domain Ω is connected, bounded and smooth so that we can invoke the Gauss-Green formula whenever needed. Then one can set $P\phi = \nabla \bar{u}$ where $\bar{u}$ solves

$$\min_{u \in H^1(\Omega)} \|\nabla u - \phi\|_{L^2(\Omega, \mathbb{R}^N)}.$$

This definition does indeed express the fact that $\nabla \bar{u}$ is the orthogonal projection of ϕ on the subspace

$$\big\{\nabla u \; ; u \in H^1(\Omega)\big\}.$$

The latter is actually a closed subspace in $L^2(\Omega, \mathbb{R}^N)$ as will be checked in the sequel (see Remark 3.2.20). It is therefore easy to see that there exists a unique function $\psi \in L^2(\Omega, \mathbb{R}^N)$ such that one has the so-called Helmholtz decomposition

$$\phi = P\phi + \psi,$$

where

$$\int_\Omega \nabla v \cdot \psi = 0, \;\; \forall v \in C^1(\bar{\Omega}). \tag{3.1.1}$$

In [149], it is proven that a function $\psi \in L^2(\Omega, \mathbb{R}^N)$ with $\operatorname{div}\psi \in L^2(\Omega)$, has a trace on the boundary of the domain Ω whenever the latter is (at least) Lipschitz continuous. The Gauss-Green formula can then be generalized to such functions. Thus, in the present case, condition (3.1.1) means that

$$\begin{cases} \operatorname{div}\psi & = 0 \text{ in } \Omega, \\ \psi \cdot \nu & = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1.2}$$

where $\psi \cdot \nu$ designates the trace of ψ at the boundary. Note that the integral in (3.1.1) makes sense even for a measure ψ, thus generalizing (3.1.2) for measures. In the end, for any $u \in H^1(\Omega)$ one can write

$$\inf_{\substack{\phi \in L^2(\Omega,\mathbb{R}^N) \\ P\phi = Du}} \int_\Omega |\phi| = \inf_{\substack{\psi \in L^2(\Omega,\mathbb{R}^N) \\ \operatorname{div}\psi = 0,\ \psi\cdot\nu = 0}} \int_\Omega |Du + \psi|.$$

Though, the integral to the right is just the total variation of $Du + \psi$ and thus still makes sense if function $u \in BV(\Omega)$ and ψ is a measure. This legitimates the following definition

$$J(u) = \inf_{\substack{\operatorname{div}\psi = 0 \\ \psi\cdot\nu = 0}} \int_\Omega |Du + \psi|.$$

The infimum is taken on all measures on Ω. To be more precise, we shall only deal with bounded Radon measures. We remark in passing that J is well-defined on $BV(\Omega)$ since one has

$$J(u) \leq \int_\Omega |Du|.$$

This functional and a variant will be the object of our study in this chapter. After some mathematical preliminaries, we are going to use a convex duality argument to show that under the appropriate assumptions on Ω (bounded and smooth are enough), one has

$$J(u) = \sup_{\substack{w \in C^1(\bar{\Omega}) \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w \cdot Du.$$

This result will then allow us to deal with sets of finite perimeter. Indeed, we shall see that for such a set E, $J(\chi_E)$ is nothing else but the perimeter $P(E, \Omega) = \int_\Omega |D\chi_E|$. This equality can actually be generalized to functions of bounded variation whose diffuse part vanishes. To prove this result, we are going to exploit some fine properties of functions of bounded variation. Though, some of them have already been mentioned and used in Chapter 2, we need to recall and detail them in this chapter.

The dual formulation will also help us discuss the possibility of proving a version of the Poincaré inequality for J.

In the last part of the present chapter, we propose a possible approach to tackle the problem of restoring a corrupted image by means of this functional. Moreover, we shall detail the algorithmic aspects of the minimization of a discretized energy involving J. Finally, we are going to compare numerically our approach with the one based on the total variation.

Parts of our results were published in [100, 101].

3.2 Preliminaries

Before getting to our main topic, we need to recall some classical results on generalized sequences, then some basic measure theory and finally some convex analysis that we are going to need in the sequel. We are also going to recall an important theorem that dates back to the work of De Rham and that will let us get back to the Helmholtz decomposition.

3.2.1 Generalized sequences

A sequence is a function whose domain is in general the set of integer numbers which is a set that is totally ordered. It is then possible to generalize this concept by allowing more general index sets: the so-called *directed sets*. This new concept is of particular interest when one deals with functional spaces that are not metrizable. Then the various sequential criteria that we all know are still valid, provided we consider generalized sequences instead of the usual sequences. This is what we are going to see in what follows. Proofs for the results we shall mention can be found in the textbooks [108, 156].

Definition 3.2.1. A set A is *oriented* if on one hand, it is equipped with a partial order $\preceq$ and on the other hand, each pair of elements of A has an upper bound in A *i.e.* whatever $a, b \in A$ there exists $m \in A$ such that $a \preceq m$ and $b \preceq m$.

Definition 3.2.2. A *generalized sequence* (or *net*) of a non-empty set X is a function $x : A \to X$ where A is oriented. By analogy with classical sequences, $x(\alpha)$ will be denoted x_α and $(x_\alpha)_{\alpha\in A}$ or simply (x_α) will denote the function $x : A \to X$ itself.

So any real function and any sequence indexed by $\mathbb{N}$ is a generalized sequence.

Definition 3.2.3. A generalized sequence $(x_\alpha)_{\alpha\in A}$ of a topological space X *converges to a limit* $x \in X$ if for any neighborhood V of x, there exists $\alpha_V \in A$

such that $x_\alpha \in V$ when $\alpha \succeq \alpha_V$. It will be said of $(x_\alpha)_{\alpha\in A}$ that it *converges* if it converges to an element of X.

This notion of limit obviously includes the one known for sequences indexed by natural numbers. The following results show that any topology can be described in terms of generalized sequences.

Theorem 3.2.4. *Let E be a subset of a topological space X. Then an element $x \in \bar{E}$ if and only if there exists a generalized sequence of E converging to x.*

Corollary 3.2.5. *Let X be a topological space and $E \subset X$. Then E is closed if and only if E contains all limits of convergent generalized sequences of its own elements.*

Corollary 3.2.6 (Semi-continuity criteria)**.** *Let X be a topological space. Then $f : X \to \mathbb{R}$ is lower semicontinuous at $x \in X$ if and only if for any generalized sequence $(x_\alpha)_{\alpha\in A}$ converging to x,*

$$f(x) \leq \liminf_{x_\alpha \to x} f(x_\alpha) = \sup_{l\in A} \inf_{\alpha \succeq l} f(x_\alpha).$$

Function $g : X \to \mathbb{R}$ is upper semicontinuous at x if and only if for any generalized sequence as above

$$g(x) \geq \liminf_{x_\alpha \to x} g(x_\alpha) = \inf_{l\in A} \sup_{\alpha \succeq l} g(x_\alpha).$$

Corollary 3.2.7 (Continuity criterion)**.** *Let X, Y be topological spaces. A function $f : X \to Y$ is continuous on X if and only if it preserves the convergence of generalized sequences of X.*

The notions of continuity and semicontinuity are essential for minimization problems. But, what can be said about the sequential compactness criterion? Does it also remain valid in this new framework? As will be seen in the next theorem, the answer is yes in some sense, but, first of all, we have to formulate it correctly in the language of generalized sequences:

Definition 3.2.8. Let $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two generalized sequences. We shall say that $(y_\beta)_{\beta\in B}$ is a *generalized subsequence* of $(x_\alpha)_{\alpha\in A}$ if for any $\alpha_0 \in A$, there exists $\beta_0 \in B$ such that

$$\{y_\beta; \beta \succeq \beta_0\} \subset \{x_\alpha; \alpha \succeq \alpha_0\}.$$

Remark 3.2.9. As for sequences indexed by $\mathbb{N}$, lower and upper limits are obtained as limits of convergent generalized subsequences.

Theorem 3.2.10 (Compactness criterion)**.** *A topological space X is compact if and only if any generalized sequence of X admits a convergent generalized subsequence.*

3.2.2 A bit of measure theory

To fix notations, recall that for a topological space X, $C_b(X)^M$ denotes the set of $\mathbb{R}^M$-valued bounded continuous functions on X and $C_0(X)^M$ the set of continuous functions with values in $\mathbb{R}^M$ that tend to 0 at infinity, *i.e.* those functions f for which, given $\varepsilon > 0$, there exists a compact of X such that $\|f\|_\infty < \varepsilon$ outside this compact. These two functional spaces will be equipped with the infinity norm. If X is equipped with the Borel σ-algebra we also recall that the Riesz representation theorem then identifies the dual of the Banach space $C_0(X)^M$ with the space $\mathcal{M}_b(X)^M$ of bounded Radon measures on X with values in $\mathbb{R}^M$. The dual norm corresponds then simply to the total variation measure. Moreover, we say that a generalized sequence $(\mu_\alpha)_{\alpha\in A}$ converges weakly-* (or in the sense $\sigma(\mathcal{M}_b(X)^M, C_0(X)^M))$ to μ if for all $f \in C_0(X)^M$ we have

$$\int_\Omega f d\mu_\alpha \to \int_\Omega f d\mu.$$

One can also consider a slightly stronger convergence: we say that a generalized sequence $(\mu_\alpha)_{\alpha\in A}$ of measures converges narrowly (or in the sense $\sigma(\mathcal{M}_b(X)^M, C_b(X)^M))$ if this time we have $\int_\Omega f d\mu_\alpha \to \int_\Omega f d\mu$ whatever $f \in C_b(X)^M$. This convergence is not a weak-* convergence in the classical sense since there is no reason *a priori* for $\mathcal{M}_b(X)^M$ to be the topological dual of $C_b(X)^M$. However, given two vector spaces E and E^* not necessarily related, it is always possible to consider that they are in duality thanks to a separating bilinear form $\langle .,. \rangle$ defined on $E \times E^*$ as far as we can define the latter. Here, "separating" means that for every $x \in E$ (respectively $x^* \in E^*$), the linear forms $\langle x, . \rangle$ (respectively $\langle ., x^* \rangle$) separate any two elements. E and E^* are respectively equipped with the finest topology that makes $\langle \cdot, x^* \rangle$ and $\langle x, \cdot \rangle$ continuous functions. Of course, one can extend the notions of weak convergence for the pair (E, E^*). Indeed, we say that a generalized sequence $(x_\alpha)_{\alpha\in A}$ of elements of E converges weakly to $x \in E$ if for every element x^* of E^* one has

$$\langle x_\alpha, x^* \rangle \to \langle x, x^* \rangle.$$

The limit is then necessarily unique. We define the weak convergence of a generalized sequence of elements of E^* similarly. Readers who wish to know more on this topic can refer to [36, 80]. In the present case, we can therefore assume that $C_b(X)^M$ and $\mathcal{M}_b(X)^M$ are in duality via the bilinear form

$$\langle \mu, f \rangle = \int_X f d\mu$$

and the "abstract" weak convergence, we have just introduced, will be exactly the narrow convergence. However, in the sequel when we talk about the weak convergence of a generalized sequence of bounded Radon measures, it will be a convergence in the sense $\sigma(\mathcal{M}_b(X)^M, C_0(X)^M)$.

In what follows, we will also need a standard weak compactness criterion we choose to recall here:

Theorem 3.2.11. *Let X be a subset of $\mathbb{R}^N$. From any bounded generalized sequence of $\mathcal{M}_b(X)^M$ one can extract a generalized subsequence converging weakly in $\mathcal{M}_b(X)^M$.*

By the criterion of compactness of the preceding paragraph, it is a simple corollary of the Banach-Alaoglu theorem. But this criterion does not say much about the narrow convergence. To state a result in this direction, we need to introduce the notion of tension:

Definition 3.2.12. A sequence $(\mu_n)_{n\in\mathbb{N}}$ of bounded Radon measures with values in $\mathbb{R}^M$ is *tight* if for any $\varepsilon > 0$, there exists a compact set K_ε such that one has $|\mu_n|(\Omega \setminus K_\varepsilon) < \varepsilon$ for any $n \in \mathbb{N}$.

Theorem 3.2.13 (Prokhorov's theorem)**.** *Let X be a subset of $\mathbb{R}^N$. Then from any sequence of $\mathcal{M}_b(X)^M$ that is bounded and tight, one can extract a subsequence converging narrowly in $\mathcal{M}_b(X)^M$.*

This theorem as well as the notion of strong convergence is mainly used by probabilists and we will not have to use it in the sequel.

The results of this section are standard and can be found with their proofs in [8].

3.2.3 Convex duality

Duality is a very important notion for convex functions. It is used in particular when one is interested in inverting an infimum and a supremum. A reference for the following results is the classical textbook of Ekeland and Temam [80].

Definition 3.2.14. Let E and E^* be two vector spaces that are in duality thanks to a separating bilinear form $\langle .,. \rangle$. We also consider a convex function $f : E \to \overline{\mathbb{R}}$. Its *Legendre conjugate* f^* is a function from E^* to $\overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x\in E} \big(\langle x^*, x\rangle - f(x)\big),$$

and its *Legendre biconjugate* $f^{**} : E \to \overline{\mathbb{R}}$ is defined by

$$f^{**}(x) = \sup_{x^*\in E^*} \big(\langle x^*, x\rangle - f^*(x^*)\big).$$

Theorem 3.2.15 (Legendre-Fenchel-Moreau Duality)**.** *Let E and E^* be two normed vector spaces in separating duality, $f : E \to \overline{\mathbb{R}}$ be a lower semicontinuous convex function. Then*

$$f^{**} = f.$$

Remark 3.2.16. The lower semicontinuity assumption is crucial here. In $\mathbb{R}^N$, a convex function is automatically continuous in the interior of its domain, but there is no guarantee that it is lower semicontinuous at the boundary of the domain.

3.2.4 A De Rham theorem and an application

If S is a distribution on an open set Ω and φ is a function lying in $C_c^\infty(\Omega)^N$ with zero divergence then it is easily checked that

$$\langle \nabla S, \varphi \rangle = \sum_{i=1}^{N} \langle \partial_i S_i, \varphi_i \rangle = -\sum_{i=1}^{N} \langle S, \partial_i \varphi_i \rangle = -\langle S, \operatorname{div} \varphi \rangle = 0.$$

The theorem that follows is the converse of this property and is due to De Rham.

Theorem 3.2.17. *Let Ω be an open set in $\mathbb{R}^N$ and T be a distribution of $\mathcal{D}'(\Omega, \mathbb{R}^N)$. It is further assumed that for any test function φ such that $\operatorname{div} \varphi = 0$, one has $\langle T, \varphi \rangle = 0$. Then there exists a distribution $S \in \mathcal{D}'(\Omega)$ such that*

$$T = \nabla S.$$

We give here another result of the same kind on the L^2 regularity of a distribution and that reminds us of the Poincaré theorem:

Theorem 3.2.18. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and S a distribution in $\mathcal{D}'(\Omega)$. Then if ∇S lies in $L^2(\Omega)^N$ one actually has $S \in H^1(\Omega)$.*

Let us now see a first application of these two results. In particular, this will allow us to get back to the Helmholtz decomposition we mentioned in the introduction and clarify some results. To begin with, given an open set Ω with Lipschitz continuous boundary, consider the following three spaces:

$$\begin{aligned} E &= \left\{ u \in L^2(\Omega)^N \ ; \ \operatorname{div} u \in L^2(\Omega) \right\} \\ V &= \left\{ u \in C_c^\infty(\Omega)^N \ ; \ \operatorname{div} u = 0 \right\} \\ H &= \text{closure of } V \text{ in } L^2(\Omega)^N \end{aligned}$$

Note that E is a Hilbert space when equipped with the inner product

$$\langle u, v\rangle_E = \langle u, v\rangle_{L^2(\Omega)^N} + \langle \operatorname{div} u, \operatorname{div} v\rangle_{L^2(\Omega)}.$$

Let us verify that the associated norm is complete. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in E, then it is Cauchy in $L^2(\Omega)^N$ so it converges to $u \in L^2(\Omega)^N$. Similarly, $(\operatorname{div} u_n)_{n\in\mathbb{N}}$ converges to $v \in L^2(\Omega)$. Necessarily, $v = \operatorname{div} u$ and as a consequence $(u_n)_{n\in\mathbb{N}}$ converges to $u \in E$.

One also recalls that there exists a trace operator $\gamma_\nu \in \mathcal{L}\left(E, L^2(\partial\Omega)\right)$ for which the Gauss-Green formula remains valid. For a function $u \in E$ that is also C^1 near the boundary, the previous trace coincides with the classical notion of trace *i.e.* $\gamma_\nu(u) = u \cdot \nu_{|\partial\Omega}$. This is the reason why, in the sequel, the trace of u will be denoted $u \cdot \nu$ even though u does not lie in $C^1(\bar{\Omega})$.

The aforementioned results can be found in one form or another in [71, 149]. In particular, one can find sketches of proofs for the two theorems that we mentioned.

Now we are ready to state the following important theorem:

Theorem 3.2.19. *Let Ω be a bounded open set in $\mathbb{R}^N$ with a Lipschitz boundary. Then,*

(i) $H^\perp = \left\{\nabla u \ ; \ u \in H^1(\Omega)\right\}$,

(ii) $H = \left\{u \in L^2(\Omega)^N ; \ \operatorname{div} u = 0, \ u \cdot \nu = 0\right\}$,

(iii) $L^2(\Omega)^N = H \oplus H^\perp$.

Proof. Let us prove (i). Let $u \in H^1(\Omega)^N$, then for any $v \in V$,

$$\langle \nabla u, v\rangle = -\langle u, \operatorname{div} v\rangle = 0$$

and as a consequence $\nabla u \in H^\perp$. Conversely, one has to prove that any $w \in H^\perp$ is the gradient of a function $H^1(\Omega)$. This follows by applying Theorems 3.2.17 and 3.2.18 successively.

Now let us turn to (ii) and denote H' the set to the right. We will first prove the inclusion $H \subset H'$. So let $u \in H$, then by definition, $u = \lim_{n\to\infty} u_n$ where $(u_n)_{n\in\mathbb{N}}$ is a sequence in V. This convergence in $L^2(\Omega)^N$ implies a convergence in the sense of distributions and since derivation is a continuous operator in the space of distributions $\operatorname{div} u_n = 0$ implies that $\operatorname{div} u = 0$. As a consequence, $(u_n)_{n\in\mathbb{N}}$ converges actually to u in E. The trace operator enjoys a good notion of continuity on E, $u \cdot \nu = \lim_{n\to\infty} u_n \cdot \nu = 0$. Thus $u \in H'$. Conversely, consider the orthogonal H'' to H in H'. By (i), an element of H'' is of the form ∇u where

$u \in H^1(\Omega)$, and satisfies in addition

$$\begin{cases} \Delta u = \operatorname{div} \nabla u = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = \nabla u \cdot \nu = 0 \text{ on } \partial\Omega. \end{cases}$$

This implies that u is constant in each connected component of Ω hence $\nabla u = 0$. Consequently, $H'' = \{0\}$ and $H = H'$.

As for (iii), one proved in the introduction that any function $\phi \in L^2(\Omega)^N$ that has a decomposition of the form

$$\phi = \nabla u + \psi$$

where $\psi \in H$ and $\nabla u \in H^{\perp}$. $\square$

Remark 3.2.20. We would like to point out that item (i) establishes the fact that $\{\nabla u \ ; \ u \in H^1(\Omega)\}$ is a closed set in $L^2(\Omega)^N$. This was left aside in the introduction.

3.3 Definition and first properties

In this section, we are going to derive important properties for the functional J. We are also going to introduce a variant that is of some interest.

3.3.1 Definition

Let $\Omega \subset \mathbb{R}^N$ be an open set then, for any $u \in BV_{loc}(\Omega)$, we set

$$J(u,\Omega) = \inf\left\{ \int_\Omega |Du + \psi| \ / \ \psi \in \mathcal{M}_b(\Omega)^N \text{ and } \int_\Omega \nabla v \cdot \psi = 0 \ \forall v \in C^1(\bar{\Omega}) \right\},$$
$$\tilde{J}(u,\Omega) = \inf\left\{ \int_\Omega |Du + \psi| \ / \ \psi \in \mathcal{M}_b(\Omega)^N \text{ and } \int_\Omega \nabla v \cdot \psi = 0 \ \forall v \in C^1_c(\Omega) \right\}.$$

We might simply denote $J(u)$ and $\tilde{J}(u)$ when there is no ambiguity. We recall that for a smooth ψ, the condition

$$\int_\Omega \nabla v \cdot \psi = 0, \ \forall v \in C^1(\bar{\Omega})$$

is equivalent to the fact that $\operatorname{div} \psi = 0$ in Ω and $\psi \cdot \nu = 0$ on $\partial\Omega$, whereas

$$\int_\Omega \nabla v \cdot \psi = 0, \ \forall v \in C^1_c(\Omega)$$

merely enforces $\operatorname{div}\psi = 0$ in Ω. Thus, J differs from $\tilde{J}$ by a Neumann type constraint at the boundary.

Let us collect in the following proposition some basic consequences of the previous definition:

Proposition 3.3.1. *Let Ω an open set of $\mathbb{R}^N$, then one has:*

(i) For any $u \in BV(\Omega)$,

$$\tilde{J}(u) \leq J(u) \leq \int_\Omega |Du|.$$

thus both J and $\tilde{J}$ define semi-norms on the space $BV(\Omega)$.

(ii) If $u \in BV(\Omega)$, in the definition of $\tilde{J}(u)$ the infimum is attained that is to say

$$\tilde{J}(u) = \min_{\operatorname{div}\psi=0} \int_\Omega |Du + \psi|.$$

(iii) If we denote $\mathcal{H}(\Omega)$ the space of harmonic functions in Ω then

$$\{\tilde{J} = 0\} = \mathcal{H}(\Omega).$$

Proof. Let us prove item (ii). Consider a minimizing sequence $(\psi_n)_{n\in\mathbb{N}}$ that satisfies

$$\tilde{J}(u) \leq \int_\Omega |Du + \psi_n| \leq \tilde{J}(u) + \frac{1}{n}.$$

This sequence is in fact bounded since

$$\limsup_{n\to\infty} \int_\Omega |\psi_n| \leq \limsup_{n\to\infty} \left(\int_\Omega |Du| + \tilde{J}(u) + \frac{1}{n} \right) \leq 2 \int_\Omega |Du|.$$

As a consequence, up to extraction of a subsequence, it converges to some $\psi_u \in \mathcal{M}_b(\Omega)^N$ and one can verify that the condition on the divergence remains true at the limit. Indeed, for any function $v \in C_c^1(\Omega)$, one has

$$\int_\Omega \nabla v \cdot \psi_n \to \int_\Omega \nabla v \cdot \psi.$$

Finally, the semicontinuity of the total variation ensures that

$$\int_\Omega |Du + \psi_u| \leq \liminf_{n\to\infty} \int_\Omega |Du + \psi_n| \leq \tilde{J}(u).$$

Let us now turn to (iii). Given a harmonic function u, one has $\nabla u + \psi = 0$ for the choice $\psi = -\nabla u$ which is admissible since $\operatorname{div}(\psi) = 0$. Conversely, from

(ii) there exists $\psi_u \in \mathcal{M}_b(\Omega)^N$ with $\mathrm{div}(\psi_u) = 0$ such that

$$\tilde{J}(u) = \int_\Omega |Du + \psi_u| = 0.$$

Hence $\Delta u = \mathrm{div}(Du) = 0$. □

3.3.2 Lower semicontinuity of $\tilde{J}(u)$

Theorem 3.3.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then $\tilde{J}$ is convex and*

$$Du \mapsto \tilde{J}(u)$$

is lowersemicontinuous for $\sigma(\mathcal{M}_b(\Omega)^N, C_0(\Omega)^N)$.

We are going to give two proofs for this theorem. The first one uses the notion of generalized sequence introduced previously.

Proof 1 of Theorem 3.3.2. Given a bounded Radon measure μ, let us set

$$\tilde{H}(\mu) = \inf_{\mathrm{div}\,\psi=0} \int_\Omega |\mu + \psi|.$$

Then $\tilde{H}$ is a function of $\mathcal{M}_b(\Omega)^N \to \mathbb{R}$ and for any $u \in BV(\Omega)$, one simply has $\tilde{H}(Du) = \tilde{J}(u)$. Note that, one knows from the proof of Proposition 3.3.1, that, in the definition of $\tilde{H}$, the infimum is attained. On the other hand, $\tilde{H}$ is obviously convex because the function of two variables $(\mu, \psi) \longmapsto \int_\Omega |\mu + \psi|$ is.

It remains to show that $\tilde{H}$ is lower-semicontinuous. Given the semicontinuity criterion of Section 3.2.1, it is sufficient to prove that for any generalized sequence $(\mu_\alpha)_{\alpha\in A}$ converging weakly to $\mu \in \mathcal{M}_b(\Omega)^N$, one has

$$\tilde{H}(\mu) \leq \liminf_{\mu_\alpha \to \mu} \tilde{H}(\mu_\alpha).$$

Let us consider a generalized subsequence still denoted $(\mu_\alpha)_{\alpha\in A}$ that minimizes the right hand side (see Remark 3.2.9). Thus, we have to prove that

$$\tilde{H}(\mu) \leq \lim_{\mu_\alpha \to \mu} \int_\Omega |\mu_\alpha + \psi_{\mu_\alpha}|.$$

Obviously, we can assume that the limit is finite. So by the corollary of the Banach-Alaoglu theorem, $(\mu_\alpha + \psi_{\mu_\alpha})_{\alpha\in A}$ converges weakly up to extraction of a subsequence. Now, let us not forget that we have assumed that the generalized sequence $(\mu_\alpha)_{\alpha\in A}$ converges weakly to μ so, the limit being unique, $(\psi_{\mu_\alpha})_{\alpha\in A}$ converges weakly to some ψ_μ and the condition on the divergence remains valid at the limit *i.e.* $\operatorname{div}\psi_\mu = 0$. Then, by lower-semicontinuity of the total variation, one has

$$\tilde{H}(\mu) \leq \int_\Omega |\mu + \psi_\mu| \leq \liminf_{\mu_\alpha \to \mu} \int_\Omega |\mu_\alpha + \psi_{\mu_\alpha}| = \liminf_{\mu_\alpha \to \mu} \tilde{H}(\mu_\alpha),$$

which proves the claim. □

The previous proof, though simple uses the notion of generalized sequences. We decide to provide an alternative proof for the lower-semicontinuity where we are going to classical sequences.

Proof 2 of Theorem 3.3.2. Step 1: We use the notations introduced in the beginning of the previous proof. To prove the lower semicontinuity, one would like to show that if

$$\ell = \liminf_{\mu} \tilde{H}(\mu) = \sup_{V \ni \mu} \inf_{V} \tilde{H}(\mu)$$

then $\ell \geq \tilde{H}(\mu)$. Note that the supremum above is taken over (weak-*) open sets V that contain μ. If $(V_k)_{k\in\mathbb{N}}$ is a sequence of open sets that contain μ, then

$$\ell \geq \sup_k \inf_{V_k} \tilde{H}.$$

Then one can find ν_k such that $\ell \geq \tilde{H}(\nu_k)$, with $\nu_k \in V_k$ for any $k \in \mathbb{N}$. The problem is that, *a priori*, ν_k does not have a limit. On the other hand, one knows from the proof of Proposition 3.3.1 that, for any $\mu \in \mathcal{M}_b(\Omega)^N$,

$$\tilde{H}(\mu) = \inf_{\operatorname{div}\psi=0} \int_\Omega |\mu + \psi| = \int_\Omega |\mu + \psi_\mu|,$$

for some measure ψ_μ with zero divergence. The limit above can thus be written

$$\ell \geq \limsup_k \int_\Omega |\nu_k + \psi_{\nu_k}|.$$

Now, if one sets $\lambda_k = \nu_k + \psi_{\nu_k}$, we get a sequence of measures that is bounded and thus converges weakly to a measure λ. Hence $\ell \geq \int_\Omega |\lambda|$.

Suppose that one is able to prove that for some appropriate neighborhoods V_k, one has $\operatorname{div}\lambda = \operatorname{div}\mu$ (that is to say, $\operatorname{div}(\lambda - \mu) = 0$). Then we would get

$$\tilde{H}(\mu) \leq \int_\Omega |\mu + (\lambda - \mu)| = \int_\Omega |\lambda| \leq \ell,$$

that is the lower-semicontinuity that we had to prove.

Step 2: **Construction of V_k.** Let us first admit that there exists a sequence

$(w_l)_{l\in\mathbb{N}}$ of functions of class $C_c^1(\Omega)$, that is countable and dense in the sense that for any function $w \in C_c^1$ and any $\varepsilon > 0$, there exits an $l \in \mathbb{N}$ such that $\|\nabla w - \nabla w_l\|_\infty \leq \varepsilon$. Let us then set for any integer k,

$$V_k = \bigcap_{l\leq k} \left\{ \nu, \int_\Omega \nabla w_l \cdot \nu - \int_\Omega \nabla w_l \cdot \mu < \frac{1}{k} \right\}.$$

These are weak-* open sets that contain μ. For any $l \in \mathbb{N}$, if $k \geq l$ then, given that $\operatorname{div} \psi_{\nu_k} = 0$, one can see that

$$\int_\Omega \nabla w_l \cdot (\lambda_k - \mu) = \int_\Omega \nabla w_l \cdot (\nu_k - \mu) < \frac{1}{k}$$

and, passing to the limit, we get

$$\int_\Omega \nabla w_l \cdot (\lambda - \mu) \leq 0.$$

Now, if $w \in C_c^1(\Omega)$ and $\varepsilon > 0$, given that there exists $l \in \mathbb{N}$ such that $\|\nabla w - \nabla w_l\|_\infty \leq \varepsilon$, one has

$$\int_\Omega \nabla w \cdot (\lambda - \mu) \leq \int_\Omega \nabla w_l \cdot (\lambda - \mu) + \|\nabla w - \nabla w_l\|_\infty \int_\Omega |\lambda - \mu| \leq \varepsilon \int_\Omega |\lambda - \mu|,$$

and, ε being arbitrarily small, it follows

$$\int_\Omega \nabla w \cdot (\lambda - \mu) \leq 0.$$

Thus replacing w by $-w$, we get $= 0$. This implies that $\operatorname{div}(\lambda - \mu) = 0$. As a consequence, H is lower-semicontinuous, provided one can find the w_l).

Step 3: **Construction of w_l.** For any $n \in \mathbb{N}$, let $\Omega_n = \{x \in \Omega,\ d(x, \partial\Omega) > 1/n\}$. By the Stone-Weierstrass theorem, we know that it is possible to find, for any $n \in \mathbb{N}$, a sequence $(v_k^n)_{k\in\mathbb{N}}$ of smooth functions, with support in $\bar{\Omega}_n$, dense in the set of functions with the same support.

Consider ρ a regularizing kernel (that we assume to be symmetric, positive, of class C^∞, with compact support in the unit ball and such that $\int_{R^N} \rho = 1$), and let us denote $\rho_m(x) = m^N \rho(mx)$. Consider all the functions $w_{k,m,n} = \rho_m * v_k^n$, for any $n \in \mathbb{N},\ k \in \mathbb{N}$ and $m > n$, so that they are of compact support.

Let $w \in C_c^1(\Omega)$ and $\varepsilon > 0$. Pick $n \in \mathbb{N}$ such that the support of w is in Ω_n. Then we know that $\rho_m * \nabla w \to \nabla w$ uniformly when $m \to +\infty$, so if m is great enough,

$$\|\rho_m * \nabla w - \nabla w\|_\infty \leq \varepsilon.$$

Now, there exists a k such that

$$\|v_k^n - w\|_\infty \leq \frac{\varepsilon}{\int_{B_{1/m}|\nabla\rho_m|}}.$$

Then, it follows

$$\begin{aligned}
\|\nabla w_{k,m,n} - \nabla w\|_\infty &\leq \|\nabla w_{k,m,n} - \rho_m * \nabla w\|_\infty + \|\rho_m * \nabla w - \nabla w\|_\infty \\
&\leq \|(\nabla\rho_m) * (v_k^n - w)\|_\infty + \varepsilon \\
&\leq \|v_k^n - w\|_\infty \|\nabla\rho_m\|_1 + \varepsilon \\
&\leq 2\varepsilon
\end{aligned}$$

which is exactly what we wanted. So, we just need to choose $(w_l)_{l\in\mathbb{N}}$ with an appropriate renumbering of the countable family $(w_{k,m,n})_{n\in\mathbb{N},\ k\in\mathbb{N},\ m>n}$. □

Remark 3.3.3. This second proof may look tedious but we have to keep in mind that the space of bounded Radon measures $\mathcal{M}_b(\Omega)$ equipped with the weak-* topology is not metrizable. This forced us either to use generalized sequences (Proof 1) or work a little harder (Proof 2). If from the beginning, we had considered a bounded set of $\mathcal{M}_b(\Omega)$ (the probability measures for instance), we could have simply used the classical sequential criteria, the weak-* topology being then metrizable (see [42]).

3.3.3 Dual formulation for $\tilde{J}(u)$

In this section, we are going to prove a dual definition for the functional $\tilde{J}$, namely:

Theorem 3.3.4 (Dual formulation of $\tilde{J}(u)$)**.** *Let $\Omega \subset \mathbb{R}^N$ open and $u \in BV(\Omega)$. Then*

$$\tilde{J}(u) = \sup_{\substack{w\in C^1(\Omega) \\ \nabla w \in C_c(\Omega)^N \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w \cdot Du.$$

If in addition, $\partial\Omega$ is connected then one simply has

$$\tilde{J}(u) = \sup_{\substack{w\in C_c^1(\Omega) \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w \cdot Du.$$

Remark 3.3.5. Note that it is sufficient to require that the functions w are of class C^1 in a neighborhood of the support of the measure Du.

Proof. The proof is based on a convex duality argument. Note that we keep the notations introduced in the previous section. Since $\tilde{H}$ is convex and lower semicontinuous, Legendre-Fenchel-Moreau's duality theorem tells us that $\tilde{H}^{**} = \tilde{H}$. It remains to compute the Legendre biconjugate. Let us start with the Legendre transform of $\tilde{H}$. For any $u \in C_0(\Omega)$,

$$\begin{aligned}
\tilde{H}^*(u) &= \sup_{\mu \in \mathcal{M}_b(\Omega)^N} \left(\int_\Omega u d\mu - H(\mu) \right) \\
&= \sup_{\substack{\mu,\psi \in \mathcal{M}_b(\Omega)^N \\ \operatorname{div}\psi = 0}} \left(\int_\Omega u d\mu - \int_\Omega |\mu + \psi| \right) \\
&= \sup_{\substack{\psi \in \mathcal{M}_b(\Omega)^N \\ \operatorname{div}\psi = 0}} \left(\sup_{\mu \in \mathcal{M}_b(\Omega)^N} \left(\int_\Omega u d(\mu + \psi) - \int_\Omega |\mu + \psi| \right) - \int_\Omega u d\psi \right).
\end{aligned}$$

For the moment, let us focus on

$$\sup_{\mu \in \mathcal{M}_b(\Omega)^N} \left(\int_\Omega u d\mu - \int_\Omega |\mu| \right). \tag{3.3.1}$$

If $|u(x_0)| > 1$ at some point $x_0 \in \Omega$ then by continuity $\|u\|_\infty > 1$ in a neighborhood of x_0. If one picks a measure μ, with support in this neighborhood, one has

$$\left(\int_\Omega u d\mu - \int_\Omega |\mu| \right) \neq 0.$$

Then, choosing $\mu' = \lambda\mu$ for some $\lambda \to +\infty$ shows that the suprememum (3.3.1) has to be infinite. Moreover, if $\|u\|_\infty \leq 1$ then the supremum is clearly zero. Hence,

$$\tilde{H}^*(u) = \begin{cases} \sup\limits_{\substack{\psi \in \mathcal{M}_b(\Omega)^N \\ \operatorname{div}\psi = 0}} \displaystyle\int_\Omega u d\psi & \text{if } \|u\|_\infty \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Reasoning as above, we claim that

$$\sup_{\substack{\psi \in \mathcal{M}_b(\Omega)^N \\ \operatorname{div}\psi = 0}} \int_\Omega u d\psi$$

is either infinite or zero. If it is zero, then De Rham's theorem asserts that $u = \nabla w$ for some $w \in C^1(\Omega)$. Finally,

$$\tilde{H}^*(u) = \begin{cases} 0 & \text{if } \|u\|_\infty \leq 1 \text{ and } u = \nabla w \text{ for some } w \in C^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Now we can turn to the calculation of the Legendre biconjugate.
For $\mu \in \mathcal{M}_b(\Omega)^N$,

$$\tilde{H}^{**}(\mu) = \sup_{u \in C_0(\Omega)} \left(\int_\Omega u d\mu - \tilde{H}^*(u) \right) = \sup_{\substack{w \in C^1(\Omega) \\ \nabla w \in C_c(\Omega)^N \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w d\mu.$$

This yields the conclusion given that $\tilde{J}(u) = \tilde{H}(Du)$ for any $u \in BV(\Omega)$. $\square$

The following consequence of Theorem 3.3.4 will be useful for the direct method to apply in Section 3.6:

Corollary 3.3.6 (L^2 lower-semicontinuity)**.** *Let $\Omega \subset \mathbb{R}^N$ open. Then $\tilde{J}$ has a lower-semicontinuous extension to $L^2(\Omega)$.*

Proof. This is simply because for any $u \in BV(\Omega)$ one can write

$$\tilde{J}(u) = \sup_{\substack{w \in C^\infty(\Omega) \\ \nabla w \in C_c^\infty(\Omega)^N \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w \cdot Du = \sup_{\substack{w \in C^\infty(\Omega) \\ \nabla w \in C_c^\infty(\Omega)^N \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega u \Delta w.$$

and the rightmost term makes sense for any $u \in L^2(\Omega)$. $\square$

3.3.4 Dual formulation for $J(u)$

Theorem 3.3.7 (Dual formulation of $J(u)$)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class C^1. Then,*

$$Du \mapsto J(u)$$

is lower-semicontinuous for $\sigma(\mathcal{M}_b(\bar{\Omega})^N, C(\bar{\Omega})^N)$ and for any $u \in BV(\Omega)$,

$$J(u) = \sup_{\substack{w \in C^1(\bar{\Omega}) \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w \cdot Du.$$

Before getting further, let us make some comments:

Remarks 3.3.8.(i) Let $u \in H^1(\Omega)$ such that $\nabla u \in BV(\Omega)^N$. Note the space of functions that satisfy the latter condition were studied thouroughly in [76] (see also Remark 2.4.18). Using the Gauss-Green formula for BV functions (see [65, 66]), on can write

$$J(u) = \sup_{\substack{w \in C^1(\bar{\Omega}) \\ \|\nabla w\|_\infty \leq 1}} \left(\int_\Omega w \Delta u - \int_{\partial\Omega} w \frac{\partial u}{\partial \nu} \right).$$

One assumes that Ω is an open set for which the Gauss-Green formula is valid (for instance, Ω bounded with Lipschitz continuous boundary is fine). If, in addition, Ω is convex the then condition $\|\nabla w\|_\infty \leq 1$ is equivalent to w being Lipschitz continuous in Ω with $\|w\|_{C^{0,1}} \leq 1$. As a consequence, if one considers the Hahn decomposition (μ^+, μ^-) associated to the measure $\mu = \Delta u - \frac{\partial u}{\partial \nu}_{|\partial\Omega}$, one has

$$J(u) = W_1(\mu^+, \mu^-)$$

where

$$W_1(\mu^+, \mu^-) = \sup_{\|w\|_{\mathrm{Lip}} \leq 1} \int_\Omega w d(\mu^+ - \mu^-)$$

is the Kantorovitch-Rubinstein distance defined for positive finite measures. This distance, sometimes also called Wasserstein distance, is very important in the theory of optimal transportation. For more on this topic, we refer to [153, Chapters 1 and 7]. Let us point out that the previous theorem proves that

$$W_1(\mu^+, \mu^-) = \inf \left\{ \int_\Omega |\phi| \ ; \ \phi \in \mathcal{M}_b(\Omega)^N, \ \operatorname{div} \phi = \Delta u, \ \phi \cdot \nu = \frac{\partial u}{\partial \nu} \right\}.$$

This gives an alternative definition for this distance (see [153, 154]).

(*ii*) The fact that Ω is of class C^1 is necessary if one wishes to invoke the forthcoming approximation Lemma 3.3.10. Nonetheless, we suspect that this assumption is superfluous. Getting rid of it is certainly possible but would demand lot of technicalities as one can see it in the proof of Kantorovitch's theorem in [153]. Indeed, Villani proves the latter theorem for a compact Ω and deduces the general result by a truncation argument.

(*iii*) As already noted in Remark 3.3.5, w may not be smooth outside of $\operatorname{supp}(Du)$

To prove Theorem 3.3.7, we need some intermediate results that we are going to establish here. Let us start with a deformation lemma:

Lemma 3.3.9 (Deformation lemma)**.** *Let $\Omega \subset \mathbb{R}^N$ be a C^1 open set with compact boundary. Then for $\varepsilon > 0$ small enough, there exists a vector field $\eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that*

(*i*) *$\eta(x) = 0$ on $\{x \in \Omega, d(x, \partial\Omega) > \varepsilon\}$,*

(*ii*) *$\{x + \delta\eta(x), x \in \Omega\} \subset\subset \Omega$ for any $\delta > 0$ sufficiently small.*

Proof. To begin with, we have to localize the problem. To do so, let us consider a finite covering of $\partial\Omega$ by a family of balls $(B_i = B(x_i, r_i))_{i=1,\ldots,n}$ that are all centered at the boundary $\partial\Omega$. Since Ω is smooth, one can pick these balls in such a way that $\Omega \cap B_i$ is a C^1 subgraph. Taking if necessary smaller balls, one can assume that the hypersurface $\partial\Omega$ has only one connected component in B_i and which is close to the tangent hyperplane at the point x_i. This is the step where we actually need the boundary to be regular.

Choose now $\varepsilon > 0$ small enough such that for $\Omega_\varepsilon = \{x \in \Omega, d(x, \partial\Omega) > \varepsilon\}$ one has

$$\bar{\Omega} \setminus \Omega_\varepsilon \subset \bigcup_{i=1}^{n} B_i.$$

For each ball B_i, one can consider a function $f_i \in C^\infty(B_i)$ that has values in $[0,1]$ and is defined by

$$f_i(x) = \begin{cases} 1 & \text{if } x \in B_i \setminus \Omega, \\ 0 & \text{if } x \in B_i \cap \overline{\Omega_\varepsilon}. \end{cases}$$

Then, given a partition of unity $(g_i)_{i=1,\dots,n}$ that is subordinate to the open cover $(B_i)_{i=1,\dots,n}$ and an element $x \in \mathbb{R}^N$, one sets

$$\eta(x) = \sum_{i=1}^{n} -f_i(x) g_i(x) \nu(x_i)$$

where $\nu(x_i)$ is the outer normal vector to Ω at x_i. For any $x \notin B_i$, the product $f_i(x)g_i(x)$ is set to 0 so there is no definition problem. This way we defined a vector field that has a compact support in $\bigcup_{i=1}^{n} B_i$ and that vanishes on Ω_ε.

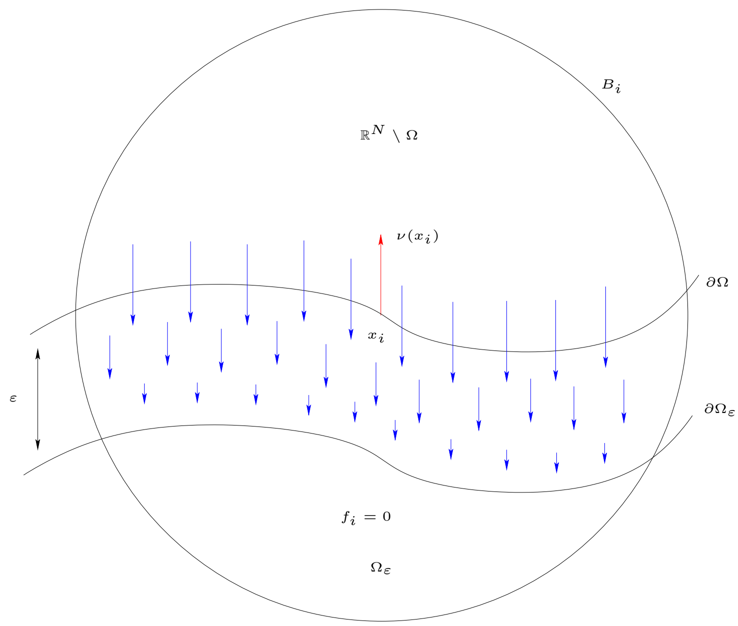


Figure 3.1: Vectorfield $-f_i\nu(x_i)$

Moreover, $I + \delta\eta$ with $0 < \delta << \varepsilon$ is a homeomorphism from Ω to its image $\{x + \delta\eta(x), x \in \Omega\}$. By the way, since we supposed that the set $\partial\Omega \cap B_i$ is close to the tangent hyperplane at x_i and since $\|\eta\|_\infty \leq 1$, the latter set is relatively compact in Ω. □

The previous lemma helps us proving the following approximation result:

Lemma 3.3.10 (Approximation lemma). *Let $\Omega \subset \mathbb{R}^N$ be a C^1 open set with compact boundary. Given a measure $\psi \in \mathcal{M}_b(\bar{\Omega})^N$ with zero divergence and $\varepsilon > 0$ small enough, there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of $\mathcal{M}_b(\Omega)^N$ such that*

$$\phi_n \to \psi \text{ in the sense } \sigma(\mathcal{M}_b(\bar{\Omega})^N, C_0(\bar{\Omega})^N).$$

Moreover,

(*i*) $\operatorname{div} \phi_n = 0$ *in* Ω,

(*ii*) $\phi_n = 0$ *in a neighborhood of* $\partial\Omega$,

(*iii*) $\phi_n = \psi$ *in* $\Omega_\varepsilon = \{x \in \Omega, d(x, \partial\Omega) > \varepsilon\}$,

(*iv*) $\int_\Omega |\phi_n| \to \int_{\bar{\Omega}} |\psi|$.

In other words, one can approximate ψ by zero divergence measures that coincide with ψ inside Ω. Furthermore, one demands convergence of the total variations.

Proof. Take ε, δ and η as in the previous lemma. Recall that one can let ε and δ be as small as one wishes. Let us first introduce some notations. In the sequel, A_δ will respresent the set $\{x + \delta\eta(x), x \in \Omega\}$. We are also going to consider the vector field $V_\delta : \Omega \to A_\delta$ defined by

$$V_\delta(x) = x + \delta\eta(x).$$

And finally, if $f \in C(\Omega)$ or $C(\Omega)^N$, f_δ will denote function $f \circ {V_\delta}^{-1}$.

The idea of the proof is to construct a sequence $(\phi_\delta)_{\delta>0}$ of the form

$$\phi_\delta = \begin{cases} M_\delta(V_{\delta\#}\psi) & \text{in } A_\delta, \\ 0 & \text{in } \Omega \setminus A_\delta, \end{cases}$$

where M_δ is an endomorphism of $\mathcal{M}_b(A_\delta)^N$ that we have to determine. Since $V_{\delta\#}\psi$ is a measure of $\mathcal{M}_b(A_\delta)^N$, ϕ_δ is well-defined and belongs to $\mathcal{M}_b(\Omega)^N$. Now it remains to find M_δ.

Let us start by writing that $\operatorname{div} \psi = 0$ *i.e.* for any $w \in C^1_c(\Omega)$,

$$\int_\Omega \nabla w d\psi = 0.$$

On the other hand,

$$\begin{aligned}\int_\Omega \nabla w d\psi &= \int_{A_\delta} (\nabla w)_\delta d(V_{\delta\#}\psi) \\ &= \int_{A_\delta} \big(I - \delta D\eta(x)\big)^T \nabla w_\delta(x) d(V_{\delta\#}\psi)(x).\end{aligned}$$

For any $f \in C_c(\Omega)^N$ and for any $x \in \Omega$, one sets

$$(T_\delta f)(x) = \big(I - \delta D\eta(x)\big)^T f(x).$$

Since η is of class C^∞, T_δ is a continuous operator from $C_0(A_\delta)^N \to C_0(A_\delta)^N$ and has, as a consequence, an adjoint operator $T_\delta{}^*$ that is continuous from $\mathcal{M}_b(A_\delta)^N$ to itself. We can now resume our calculation,

$$\begin{aligned}\int_{A_\delta} \big(I - \delta D\eta(x)\big)^T \nabla w_\delta(x) d(V_{\delta\#}\psi)(x) &= \int_{A_\delta} T_\delta \nabla w_\delta d(V_{\delta\#}\psi) \\ &= \int_{A_\delta} \nabla w_\delta d\big(T_\delta{}^*(V_{\delta\#}\psi)\big).\end{aligned}$$

Setting $M_\delta = T_\delta{}^*$ and putting things together, one has proven that for any $w \in C_c^1(\Omega)$,

$$\int_{A_\delta} \nabla w_\delta d\phi_\delta = 0.$$

This reflects the fact that ϕ_δ has zero divergence and thus establishes (i).

Let us get to the other items. (ii) becomes obvious if one recalls that A_δ is relatively compact in Ω (see Lemma 3.3.9).

To prove (iii), consider a function $f \in C_c^\infty(\Omega)^N$ with support in Ω_ε. As $\eta = 0$ on Ω_ε, one can write

$$\int_{\Omega_\varepsilon} f d\phi_\delta = \int_{\Omega_\varepsilon} \big(I - \delta D\eta(x)\big)^T f(x) d(V_{\delta\#}\psi)(x) = \int_{\Omega_\varepsilon} f d\psi$$

and, as a consequence, $\phi_\delta = \psi$ in Ω_ε.

It remains to prove item (iv). Let us start by the definition of the total variation:

$$\int_\Omega |\phi_\delta| = \int_{A_\delta} |\phi_\delta| = \sup_{\substack{f \in C_c^\infty(A_\delta)^N \\ \|f\|_\infty \le 1}} \int_{A_\delta} f d\phi_\delta.$$

For such an f, one has

$$\begin{aligned}\int_{A_\delta} f d\phi_\delta &= \int_\Omega \big(I - \delta D\eta\big(x + \delta\eta(x)\big)\big)^T f\big(x + \delta\eta(x)\big) d\psi(x) \\ &\leq \sup_{x \in V_\delta^{-1}(\mathrm{supp}(f))} \big\|I - \delta D\eta\big(x + \delta\eta(x)\big)\big\| \int_{A_\delta} |\psi| \\ &\leq (1 + \delta \max_{x \in \mathbb{R}^N} \|D\eta(x)\|) \int_{A_\delta} |\psi|.\end{aligned}$$

It follows that

$$\limsup_{\delta \to 0} \int_\Omega |\phi_\delta| \leq \int_{\bar\Omega} |\psi|.$$

For the moment, let us assume that we know that $\phi_\delta \rightharpoonup^* \psi$ when $\delta \to 0$. The lower-semicontinuity of the total variation then tells us

$$\int_{\bar\Omega} |\psi| \leq \liminf_{\delta \to 0} \int_\Omega |\phi_\delta|,$$

which ends the proof of (iv).

The only thing that is left is to prove that for any function $f \in C_0(\bar\Omega)$,

$$\lim_{\delta \to 0} \int_\Omega f d\phi_\delta = \int_{\bar\Omega} f d\psi.$$

This is actually true since

$$\int_\Omega f d\phi_\delta = \int_\Omega \big(I - \delta D\eta\big(x + \delta\eta(x)\big)\big)^T f\big(x + \delta\eta(x)\big) d\psi(x)$$

where the function $x \mapsto \big(I - \delta D\eta\big(x + \delta\eta(x)\big)\big)^T f\big(x + \delta\eta(x)\big)$ converges uniformly to f with $\delta \to 0$. $\square$

Proof of Theorem 3.3.7. Step 1: Let us start by considering the convex function $H : \mathcal{M}_b(\bar\Omega) \to \mathbb{R}$ defined by

$$H(\mu) = \inf_{\substack{\psi \in \mathcal{M}_b(\bar\Omega) \\ \operatorname{div} \psi = 0 \\ \psi \cdot \nu = 0}} \int_{\bar\Omega} |\mu + \psi|.$$

Our goal is to apply the convex duality theorem for H and to do so we have to prove that this function is really lsc with respect to the topology $\sigma(\mathcal{M}_b(\bar\Omega)^N, C(\bar\Omega)^N)$. Actually, the semicontinuity can be proven exactly as for the functional $\tilde H$. We thus refer to the proof of Theorem 3.3.2. Now, thanks to Theorem 3.2.15, we know that $H = H^{**}$. It remains to compute the Legendre biconjugate. Once more, it is enough to copy step by step what we did for $\tilde H^*$ in the proof of Theorem 3.3.4 to deduce that for any $u \in C(\bar{\Omega})$,

$$H^*(u) = \begin{cases} 0 & \text{if } \|u\|_\infty \leq 1 \text{ and } u = \nabla w \text{ for some } w \in C^1(\bar{\Omega}), \\ +\infty & \text{otherwise.} \end{cases}$$

It follows that the Legendre biconjugate is

$$\begin{aligned} H^{**}(\mu) &= \sup_{u \in C(\bar{\Omega})} \left(\int_{\bar{\Omega}} u d\mu - H^*(u) \right) \\ &= \sup_{\substack{w \in C^1(\bar{\Omega}) \\ \|\nabla w\|_\infty \leq 1}} \int_{\bar{\Omega}} \nabla w d\mu. \end{aligned}$$

If we put things together, we have proven so far

$$\inf_{\substack{\psi \in \mathcal{M}_b(\bar{\Omega}) \\ \operatorname{div} \psi = 0 \\ \psi \cdot \nu = 0}} \int_{\bar{\Omega}} |\mu + \psi| = \sup_{\substack{w \in C^1(\bar{\Omega}) \\ \|\nabla w\|_\infty \leq 1}} \int_{\bar{\Omega}} \nabla w d\mu.$$

Taking $\mu = Du$, one can see that the term to the right is exactly the expression we are interested in. Unfortunately, it is not clear that the first term is equal to $J(u)$. In fact, we will see in the sequel that this is true.

Step 2: Given a measure $\mu \in \mathcal{M}_b(\bar{\Omega})$ such that $\mu_{|\partial\Omega} = 0$, we are going to prove that

$$\inf_{\substack{\psi \in \mathcal{M}_b(\bar{\Omega}) \\ \operatorname{div} \psi = 0 \\ \psi \cdot \nu = 0}} \int_{\bar{\Omega}} |\mu + \psi| = \inf_{\substack{\phi \in \mathcal{M}_b(\Omega) \\ \operatorname{div} \phi = 0 \\ \phi \cdot \nu = 0}} \int_{\Omega} |\mu + \phi|$$

hence the expected result. In fact, it would suffice to prove that for a measure $\psi \in \mathcal{M}_b(\bar{\Omega})$ satisfying $\operatorname{div} \psi = 0$ and $\psi \cdot \nu = 0$, there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of $\mathcal{M}_b(\Omega)$ such that $\operatorname{div} \phi_n = 0$ and $\phi_n \cdot \nu = 0$ with

$$\int_{\Omega} |\mu + \phi_n| \to \int_{\bar{\Omega}} |\mu + \psi|.$$

This sequence has already been constructed in the previous lemma. So this is actually the step where the regularity of the boundary comes into play. Before going further, let us point out that since $\Omega = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$ with, we recall, $\Omega_\varepsilon = \{x \in \Omega, d(x, \partial\Omega) > \varepsilon\}$, then for any $\delta > 0$, there exists $\varepsilon > 0$ as small as one wishes such that $|\mu|(\Omega \setminus \Omega_\varepsilon) < \delta$. For such an ε, the approximation lemma yields a sequence $(\phi_n)_{n \in \mathbb{N}}$ satisfying the two conditions $\operatorname{div} \phi_n = 0$, $\phi_n \cdot \nu = 0$.

In addition, one can write

$$\begin{aligned}\int_{\bar\Omega} |\mu+\psi| &= \int_{\Omega_\varepsilon} |\mu+\phi_n| + \int_{\bar\Omega\setminus\Omega_\varepsilon} |\mu+\psi| \\ &= \int_{\Omega} |\mu+\phi_n| + \int_{\bar\Omega\setminus\Omega_\varepsilon} (|\mu+\psi| - |\mu+\phi_n|).\end{aligned}$$

However,

$$\int_{\bar\Omega\setminus\Omega_\varepsilon} (|\mu+\psi| - |\mu+\phi_n|) \le \int_{\bar\Omega\setminus\Omega_\varepsilon} (|\mu+\psi| + |\mu| - |\phi_n|).$$

Since, on the one hand $\phi_n = \psi$ in Ω_ε and on the other hand $\int_\Omega |\phi_n| \to \int_{\bar\Omega} |\psi|$,

$$\int_{\bar\Omega\setminus\Omega_\varepsilon} (|\mu+\psi| + |\mu| - |\phi_n|) \to \int_{\bar\Omega\setminus\Omega_\varepsilon} (|\mu+\psi| + |\mu| - |\psi|).$$

Finally, the assumption $\mu_{|\partial\Omega} = 0$ implies that

$$\begin{aligned}\int_{\bar\Omega\setminus\Omega_\varepsilon} (|\mu+\psi| + |\mu| - |\psi|) &= \int_{\Omega\setminus\Omega_\varepsilon} (|\mu+\psi| + |\mu| - |\psi|) \\ &\le 2\int_{\Omega\setminus\Omega_\varepsilon} |\mu| \le 2\delta.\end{aligned}$$

Reasoning in a similar way with $-\left(\int_{\bar\Omega} |\mu+\psi| - \int_\Omega |\mu+\phi_n|\right)$, we have actually proven that for any $\delta > 0$,

$$\limsup_{n\to\infty} \left| \int_{\bar\Omega} |\mu+\psi| - \int_\Omega |\mu+\phi_n| \right| \le 2\delta.$$

This establishes the announced convergence and ends the proof of the theorem. □

3.4 Toward a Poincaré-type inequality for J and $\tilde J$

Theorem 3.4.1. *Let $u \in BV(\Omega)$ and $(\rho_\varepsilon)_{\varepsilon>0}$ be a mollifying sequence associated to a radial regularizing kernel ρ. Then, for any $K \subset\subset \Omega$ and $\varepsilon > 0$ sufficiently small, one has*

$$\|u * \rho_\varepsilon - u\|_{L^1(K)} \le C\varepsilon \tilde J(u,\Omega) \le C\varepsilon J(u,\Omega).$$

This improves a similar inequality that is proven for the total variation in [8, Lemma 3.2], namely for ε small

$$\|u * \rho_\varepsilon - u\|_{L^1(K)} \le C\varepsilon |Du(\Omega)|. \tag{3.4.1}$$

Proof. By a regularization argument, one can assume that $u \in C^1(\Omega)$. Then,

$$\int_K |u * \rho_\varepsilon - u| = \sup_{\substack{\varphi \in C_c^\infty(K) \\ |\varphi| \leq 1}} \int_{\mathbb{R}^N} (u * \rho_\varepsilon - u)\varphi,$$

where

$$\begin{aligned}
\int_{\mathbb{R}^N} (u * \rho_\varepsilon - u)\varphi &= \int_{\mathbb{R}^N \times \mathbb{R}^N} [u(x - \varepsilon y) - u(x)]\, \rho(y)\varphi(x) dy dx \\
&= \varepsilon \int_{\mathbb{R}^N \times \mathbb{R}^N} \int_0^1 \nabla u(x - \varepsilon t y) \cdot y\rho(y)\varphi(x) dt dy dx \\
&= \int_0^\varepsilon \int_{\mathbb{R}^N \times \mathbb{R}^N} \nabla u(x - ty) \cdot y\rho(y)\varphi(x) dy dx dt \\
&= \int_0^\varepsilon \int_{\mathbb{R}^N} \nabla u(z) \cdot \left[\int_{\mathbb{R}^N} y\rho(y)\varphi(z + ty) dy\right] dz dt \\
&= \int_0^\varepsilon \int_{\mathbb{R}^N} \nabla u(z) \cdot \left[\int_{\mathbb{R}^N} \frac{\xi - z}{t^{n+1}} \tilde{\rho}\left(\frac{|\xi - z|}{t}\right) \varphi(\xi) d\xi\right]
\end{aligned}$$

where we denoted $z = x - ty$, then $\xi = z + ty$ and $\tilde{\rho} \in C_c^\infty(\mathbb{R}^+)$ such that $\tilde{\rho} = 0$ on $[1, +\infty)$ and $\tilde{\rho}(|y|) = \rho(y)$. If, for any $s \in \mathbb{R}$, we set

$$\eta(s) = \int_0^s \lambda \tilde{\rho}(\lambda) d\lambda - \int_0^1 \lambda \tilde{\rho}(\lambda) d\lambda$$

and

$$w_{\xi,t}(z) = \frac{1}{t^{n-1}} \eta\left(\frac{|\xi - z|}{t}\right)$$

then, given that $\eta'(s) = s\tilde{\rho}(s)$, one has

$$\nabla_z w_{\xi,t}(z) = \frac{\xi - z}{t^{n+1}} \tilde{\rho}\left(\frac{|\xi - z|}{t}\right).$$

So, if one sets

$$w_{\varphi,t}(z) = \int_{\mathbb{R}^N} w_{\xi,t}(z)\varphi(\xi) d\xi,$$

one finally gets

$$\int_{\mathbb{R}^N} (u * \rho_\varepsilon - u)\varphi = \int_0^\varepsilon \int_{\mathbb{R}^N} \nabla u(z) \cdot \nabla w_{\varphi,t}(z) dz dt.$$

Moreover, $w_{\varphi,t} \in C^1(\Omega)$ with $\operatorname{supp}(w_{\varphi,t}) \subset \operatorname{supp}(\varphi) + B(0, t)$.

As a consequence,

$$\int_{\mathbb{R}^N} (u * \rho_\varepsilon - u)\varphi = \tilde{J}(u) \int_0^\varepsilon \|\nabla w_{\varphi,t}\|_\infty dt.$$

hence the claim. □

A challenging question would be to know whether there exists a Poincaré inequality in this context. In [8], (3.4.1) is a key lemma to prove the Poincaré inequality for the total variation. To get such a result for J or $\tilde{J}$ there is still some work to be done. First of all, in a bounded and smooth domain $\Omega \subset \mathbb{R}^N$, the Poincaré inequality for $\tilde{J}$ should be of the form

$$\left\|u - P_{\mathcal{H}(\Omega)}(u)\right\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C\tilde{J}(u)$$

where $P_{\mathcal{H}(\Omega)}$ is the orthogonal projection on $\{\tilde{J} = 0\} = \mathcal{H}(\Omega)$ the space of square integrable harmonic functions in Ω, which is known to be closed in $L^2(\Omega)$ (see [103] for further details). The remaining ingredients for the proof to work are a compactness theorem (that is to say, a variant of Rellich's theorem) and also an extension theorem similar to [8, Proposition 3.21].

3.5 Relation with the Total Variation

Let us recall that, for any open set $\Omega \subset \mathbb{R}^N$ and function $u \in BV(\Omega)$, one has

$$\tilde{J}(u) \leq J(u) \leq \int_\Omega |Du|. \tag{3.5.1}$$

For the moment, we consider the one-dimensional case. Then we claim that the three terms above are actually equal. Indeed, we have seen that

$$\tilde{J}(u) = \sup_{\substack{w' \in C_c(\Omega) \\ \|w'\|_\infty \leq 1}} \int_\Omega w' du'.$$

However, any function $\varphi \in C_c^1(\Omega)$ has a primitive w, so

$$\int_\Omega |u'| = \sup_{\substack{\varphi \in C_c^1(\Omega) \\ \|\varphi\|_\infty \leq 1}} \int_\Omega \varphi du' \leq \tilde{J}(u).$$

In higher dimension, one can prove in a similar way that the total variation coincides with J and $\tilde{J}$ for any radial $u \in BV(\Omega)$.

Though, we cannot expect such a result to hold in general. For instance, we know that $\tilde{J}$ vanishes on harmonic functions which is not true for TV (see also the numerical simulations of Section 3.6.3). Therefore, we could ask oneself: when does equality occur? We will see in this chapter that the three terms in (3.5.1) are actually equal if u is the characteristic function of a set of finite perimeter. Then in the second part of this section, we are going to generalize this result to functions of bounded variation whose diffuse part vanishes. Our proofs rely on a localization argument that is possible thanks to a covering lemma that we need to recall. Since we are also going to use some results on the structure of the derivative of BV functions, we are going to detail them.

3.5.1 A covering lemma

Theorem 3.5.1 (Besicovitch-Vitali covering lemma)**.** *Given μ a Radon measure, $A \subset \mathbb{R}^N$ a measurable set and $\mathcal{B}$ a family of closed balls with positive radii such that, for any $x \in A$,*

$$\inf\{r, B(x,r) \in \mathcal{B}\} = 0.$$

Then there exists a countable and disjoint $(B_i)_{i\in\mathbb{N}}$ such that

$$\mu\left(A \setminus \bigcup_{i\in\mathbb{N}} B_i\right) = 0.$$

If in addition, one is given $\varepsilon > 0$, one can choose the balls B_i such that

$$\sum_{i\in\mathbb{N}} \mu(B_i) \leq \mu(A) + \varepsilon.$$

We refer to [118] for a proof of this classical result of geometric measure theory.

3.5.2 The case of sets of finite perimeter

Let us recall that given an open set $\Omega \subset \mathbb{R}^N$, a set E is of *finite perimeter in* Ω if the quantity $P(E,\Omega) = \int_\Omega |D\chi_E|$, called *perimeter*, is finite.

Regularity of sets of finite perimeter

Definition 3.5.2. Let $\Omega \subset \mathbb{R}^N$ open, E be a set of finite perimeter in Ω. Then one defines its *reduced boundary* $\partial^* E$ as the Borelian set of all points $x \in \mathbb{R}^N$ such that

(*i*) $\int_{B(x,r)} |D\chi_E| > 0$ for any $r > 0$,

(*ii*) the limit $\nu(x) = -\lim\limits_{r\to 0} \dfrac{\int_{B(x,r)} D\chi_E}{\int_{B(x,r)} |D\chi_E|}$ exists,

(*iii*) $|\nu| = 1$.

Actually, in case ∂E is a C^1 hypersurface, $\partial^* E = \partial E$, ν is exactly the outer normal vector to Ω and one has

$$D\chi_E = -\nu \mathcal{H}^{N-1}_{|\partial E}.$$

This remark motivates the first of the two following theorems. As for the second theorem, it states that any set of finite perimeter has tangent hyperplanes in a weak sense. These formulation of Theorems 3.5.3 and 3.5.4 are borrowed from [92].

Theorem 3.5.3 (De Giorgi)**.** *Let $\Omega \subset \mathbb{R}^N$ open and E be a set of finite perimeter in Ω. Then,*

(i) $\partial^* E \subset \partial E$,

(ii) $D\chi_E = -\nu \mathcal{H}^{N-1}_{|\partial^* E}$,

(iii) $|D\chi_E| = \mathcal{H}^{N-1}_{|\partial^* E}$.

Theorem 3.5.4 (De Giorgi)**.** *Let $\Omega \subset \mathbb{R}^N$ open, E be a set of finite perimeter in Ω and $x \in \partial^* E$. Given $r, \varepsilon > 0$, one sets*

$$\begin{aligned} S_{r,\varepsilon}(x) &= \{y \in B(x,r); |\langle \nu(x), y - x\rangle| \leq r\varepsilon\}, \\ T_{r,\varepsilon}(x) &= B(x,r) \setminus S_{r,\varepsilon}. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{r\to 0} r^{1-n} \int_{T_{r,\varepsilon}(x)} |D\chi_E| &= 0, \\ \lim_{r\to 0} r^{1-n} \int_{S_{r,\varepsilon}(x)} |D\chi_E| &= \omega_{N-1}, \end{aligned}$$

where ω_{N-1} is the Lebesgue measure of the unit ball of $\mathbb{R}^{N-1}$.

Equality for sets of finite perimeter

Theorem 3.5.5. *For any open set $\Omega \subset \mathbb{R}^N$ and any set E of finite perimeter in Ω, one has*

$$\tilde{J}(\chi_E) = J(\chi_E) = P(E, \Omega).$$

Proof. We recall that

$$\tilde{J}(u) = \sup_{\substack{\nabla w \in C_c(\Omega)^N \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w \cdot Du$$

for any $u \in BV(\Omega)$ and in particular for $u = \chi_E$. To prove the announced result, we can approximate $P(E, \Omega)$ by the integral $\int_\Omega \nabla w_\varepsilon \cdot D\chi_E$ where we have to pick the functions $w_\varepsilon \in C_c^1(\Omega)$ judiciously. The key point is that we have to construct them in such a way that, locally, their gradients coincide on average with the normal vector to E. In what follows μ will denote the measure $\mathcal{H}^{N-1}_{|\partial^* E}$. We also fix a real $\varepsilon \in]0, 1[$.

At first, we must find a covering of $\partial^* E$. This way we can localize the problem. Given a point $x \in \partial^* E$, Theorem 3.5.3 asserts that on the one hand

$$\int_{B(x,r)} \nu(x) \cdot \nu(y) d\mu(y) = -\int_{B(x,r)} \nu(x) \cdot D\chi_E$$

and on the other hand

$$\int_{B(x,r)} |D\chi_E| = \mu(B(x, r)).$$

So by definition of $\nu(x)$, there exists $r_0(x)$ such that for any $r < r_0(x)$,

$$\int_{B(x,r)} \nu(x) \cdot \nu(y) d\mu(y) = (1 + \eta_1(r))\, \mu\left(B(x, r)\right),$$

where $|\eta_1(r)| < \varepsilon$.

Then, let us consider the family of balls $\mathcal{B} = (B(x, r_x))_{\substack{x \in \partial^* E \\ 0 < r_x < r_0(x)}}$. As we can apply the covering lemma, there exists a countable subfamily $(B_i = B(x_i, r_i))_{i \in \mathbb{N}}$ of pairwise disjoint balls. Since the balls of $\mathcal{B}$ are as small as one wishes, one can assume that

$$\sum_{i \in \mathbb{N}} r_i^{N-1} < +\infty. \tag{3.5.2}$$

This hypothesis will be useful in the end of the proof.

Furthermore, one has

$$\mu(\partial^* E) = \sum_{i \in \mathbb{N}} \mu(B_i) = \sum_{i=1}^{n} \mu(B_i) + \eta_2$$

where one can choose the integer n in such a way that $\eta_2 < \varepsilon$.

In what follows, S_i denotes the set $S_{r_i,\varepsilon}(x_i)$ introduced in the statement of Theorem 3.5.4 and S_i' is the set $S_{r_i,\varepsilon}(x_i) + 2r_i\varepsilon\nu(x_i)$. For any $y \in B_i$, one sets

$$w_{i,\varepsilon}(y) = \begin{cases} y \cdot \nu(x_i) + r_i\varepsilon & \text{if } y \in S_i \cap (1 - 4\varepsilon)B_i, \\ -y \cdot \nu(x_i) + 3r_i\varepsilon & \text{if } y \in S_i' \cap (1 - 4\varepsilon)B_i, \\ 0 & \text{if } y \in B_i \setminus \left[(S_i \cup S_i') \cap (1 - 2\varepsilon)B_i\right]. \end{cases}$$

We represent in the following figure the latter construction

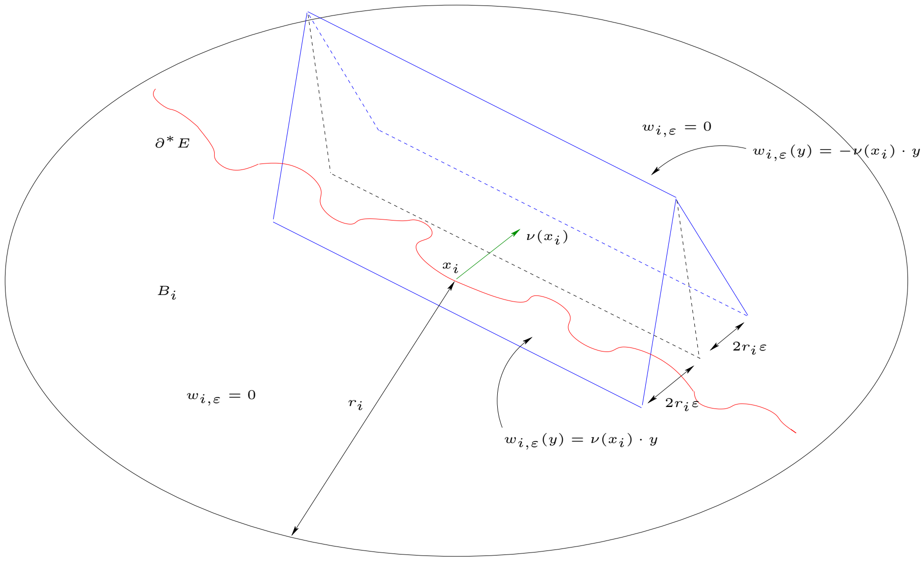


Figure 3.2: Construction of $w_{i,\varepsilon}$.

Moreover, for any $i \in \mathbb{N}$, one requires that $w_{i,\varepsilon}$ decreases regularly and that it vanishes on $(S_i \cup S_i') \setminus (1-2\varepsilon)B_i$ while keeping a gradient that satisfies $\|\nabla w_{i,\varepsilon}\|_\infty \leq 1$. The following figure will clarify the final construction:

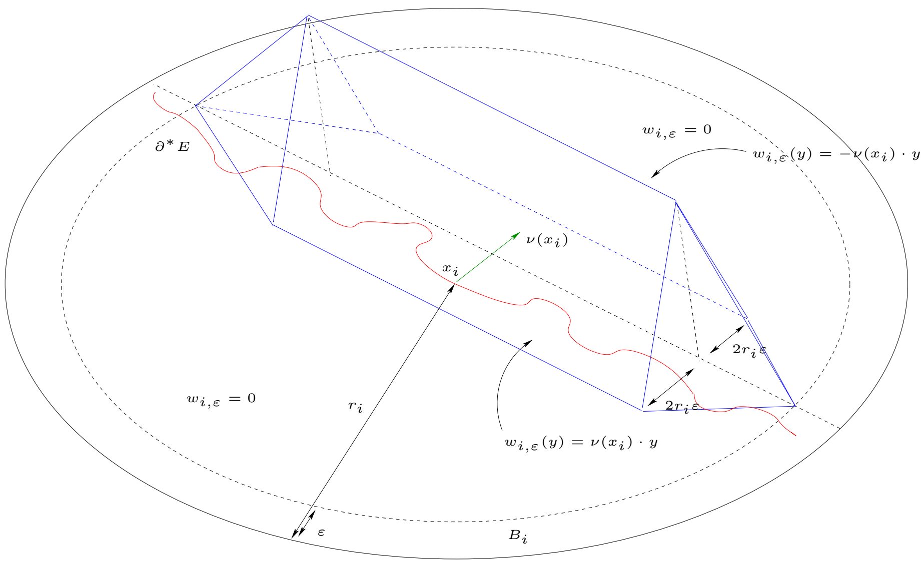


Figure 3.3: Graph of $w_{i,\varepsilon}$.

As each $w_{i,\varepsilon}$ is of compact support in B_i and since these balls were chosen to be pairwise disjoint, then one can define w_ε on Ω by

$$w_\varepsilon(y) = \begin{cases} w_{i,\varepsilon}(y) & \text{if } y \in \bigcup_{i=1}^n B_i, \\ 0 & \text{otherwise.} \end{cases}$$

One can clearly see that w_ε is Lipschitz continuous, that it has compact support in Ω and that $\|\nabla w_\varepsilon\|_\infty \leq 1$.

Then by the previous covering result, it follows

$$\begin{aligned} \int_\Omega \nabla w_\varepsilon \cdot D\chi_E &= \sum_{i=1}^n \int_{B_i} \nabla w_\varepsilon \cdot D\chi_E \\ &= \sum_{i=1}^n \left(\int_{S_i} \nu(x_i) \cdot \nu(y) d\mu(y) - \int_{S_i'} \nu(x_i) \cdot \nu(y) d\mu(y) \right) \\ &\quad - \sum_{i=1}^n \left(\int_{S_i \setminus (1-4\varepsilon)\bar{B}_i} \nu(x_i) \cdot \nu \, d\mu - \int_{S_i' \setminus (1-4\varepsilon)\bar{B}_i} \nu(x_i) \cdot \nu \, d\mu \right) \\ &\quad + \sum_{i=1}^n \int_{(1-2\varepsilon)B_i \setminus (1-4\varepsilon)\bar{B}_i} \nabla w_\varepsilon \cdot D\chi_E. \end{aligned}$$

At first, remark that thanks to Theorem 3.5.4, one could have chosen from the beginning a covering such that the general term of the first series equals

$$(1 + \eta_1(r_i)) \left(\int_{S_i} |D\chi_E| - \int_{S_i'} |D\chi_E| \right).$$

Moreover, Theorem 3.5.4 also asserts that for a radius $r_0(x_i)$ that is supposed to be small,

$$\int_{S_i} |D\chi_E| - \int_{S_i'} |D\chi_E| = (1 + \eta_3(r_i)) \, \mu(B_i),$$

with $|\eta_3(r_i)| < \varepsilon$.

As for the term that is made of the two other sums, let us denote it $\eta_4(\varepsilon)$, its absolute value is not greater than

$$C \sum_{i=1}^n \left[\mu \left(B_i \setminus (1 - 4\varepsilon)\bar{B}_i \right) \right] \leq C\varepsilon^{N-1} \sum_{i=1}^n r_i^{N-1} \leq C'\varepsilon^{N-1},$$

with a constant C' that does not depend on n, by assumption (3.5.2). So, whenever ε is small, $\eta_4(\varepsilon)$ is also small.

All in all,

$$\begin{aligned}\int_\Omega \nabla w_\varepsilon \cdot D\chi_E &= \sum_{i=1}^n (1+\eta_1(r_i))\,(1+\eta_3(r_i))\,\mu(B_i) + \eta_4(\varepsilon)\\ &\geq (1-\varepsilon)^2(\mu(\partial^* E) - \eta_2) + \eta_4(\varepsilon)\\ &\geq (1-\varepsilon)^2\left(\int_\Omega |D\chi_E| - \varepsilon\right) + \eta_4(\varepsilon).\end{aligned}$$

Making $\varepsilon \to 0$ ends the proof provided that the function w_ε is smooth. This can be achieved by regularizing and throwing away sets of μ-measure small. We can therefore safely assume that w_ε lies in $C_c^\infty(\Omega)$.

□

Remark 3.5.6. Actually, in this proof we merely used

$$\tilde{J}(u) \geq \sup_{\substack{\nabla w \in C_c(\Omega)^N \\ \|\nabla w\|_\infty \leq 1}} \int_\Omega \nabla w \cdot Du,$$

and not the duality result for $J(u)$.

Theorem 3.5.4 that we used several times in this proof hides a very interesting property of any set of finite perimeter: its boundary is $(N-1)$-rectifiable. We will get back to this notion later but let us recall its definition here:

Definition 3.5.7. Given some integer d, a set $E \subset \Omega$ is *d-rectifiable* (or simply *rectifiable* whenever $d = N-1$) if there exists a countable family of surfaces Γ_i of dimension d, where each Γ_i is, up to a change of coordinates, the graph of a C^1 function, and such that

$$\mathcal{H}^d\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

This definition suggests that the case where E of class C^1 is of particular interest. Under this assumption, it seems more natural to consider the *signed distance* $\bar{d}(x,E) = d(x,\Omega\setminus E) - d(x,E)$ instead of the function w_ε. The normal vector to E can then be obtained as the gradient of $\bar{d}(\cdot,E)$. Nonetheless, one still has to address the question of the regularity of this function. A general rule is that the regularity of the signed distance in the neighborhood of ∂E is directly related to the regularity of the boundary ∂E. In [74] (see also [9]), Delfour and Zolésio establish the equivalence for a domain whose boundary is (at least) $C^{1,1}$. They prove in particular the following result:

Theorem 3.5.8. *Let $E \subset \mathbb{R}^N$ of class $C^{1,1}$. Then for any $x \in \partial E$, there exists a neighborhood of x where $\bar{d}(\cdot, E)$ is of class $C^{1,1}$ and one has $\nabla \bar{d}(x, E) = \nu(x)$.*

The proof of this theorem is based on the fact that, locally, one has a good notion of projection on the set ∂E (*i.e.* existence of a local and unique orthogonal projection). This unfortunately fails in case the boundary ∂E is only C^1. Actually, in the same textbook the authors consider a simple domain of class $C^{1,1-1/n}$ with $n \geq 1$ such that its signed distance is not even C^1. This is the reason why we had to construct the function w_ε as we did.

3.5.3 Generalization to BV functions with vanishing diffuse part

Structure theorem for BV functions

In [8], Ambrosio, Fusco and Pallara are interested in continuity and differentiability in a weak sense of BV functions. They also give a precise description of Du that we are going to recall here. Given an open set $\Omega \subset \mathbb{R}^N$ and $u \in BV(\Omega)$, Radon-Nikodym's theorem asserts that one has a decomposition $Du = D^a u + D^s u$ where $D^a u$ is a measure that is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^N$ whereas $D^s u$ is the singular part. In fact, the absolutely continuous part can be obtained by differentiating almost everywhere *i.e.* $Du = \nabla u dx + D^s u$. As for the singular part, one can actually decompose it as $D^s u = D^j u + D^c u$ where $D^c u$ is the *Cantor part* of the derivative. In the sequel, we will forget about this part and will consider only BV functions whose *diffuse part* $D^a u + D^c u$ vanishes. As for the *jump part* $D^j u$, we recall that it is defined by restricting the measure $D^s u$ to the *jump set* J_u:

Definition 3.5.9. Let Ω be an open set of $\mathbb{R}^N$ and $u \in L^1_{\text{loc}}(\Omega)^M$. We are going to say that $x \in \Omega$ is a *jump point* of u whenever there exists $u^+(x),\ u^-(x) \in \mathbb{R}^M$ and a unitary vector $\nu(x)$ such that

$$\lim_{r \to 0} \frac{1}{|B_r^{\pm}(x, \nu(x))|} \int_{B_r^{\pm}(x,\nu(x))} |u(y) - u^{\pm}(x)| = 0$$

where $B_r^{\pm}(x, \nu(x)) = \{y \in B(x,r); \pm|\langle \nu(x), y - x\rangle| > 0\}$.
The set of jump points of u will be denoted J_u and $[u](x)$ will designate the value $u^+(x) - u^-(x)$ of the jump.

The following theorem gives a precise description of the jump part of the derivative Du:

Theorem 3.5.10 (Federer-Vol'pert). *For any $\Omega \subset \mathbb{R}^N$ open and $u \in BV(\Omega)$, J_u is an $(N-1)$-rectifiable set. Moreover,*

$$|Du|_{|J_u} = |[u]|\mathcal{H}^{N-1}_{|J_u},$$
$$Du_{|J_u} = [u]\nu\mathcal{H}^{N-1}_{|J_u}.$$

If one denotes $\nu^\perp(x)$ the hyperplane that is orthogonal to $\nu(x)$, then for any $\varphi \in C^1_c(\Omega)^N$, one has

$$\lim_{r\to 0} r^{1-N} \int_\Omega \varphi\left(\frac{y-x}{r}\right)|Du|(y) = |[u](x)| \int_{\nu(x)^\perp} \varphi(y) d\mathcal{H}^{N-1}(y),$$
$$\lim_{r\to 0} r^{1-N} \int_\Omega \varphi\left(\frac{y-x}{r}\right)\cdot Du(y) = [u](x)\nu(x) \cdot \int_{\nu(x)^\perp} \varphi(y) d\mathcal{H}^{N-1}(y),$$

for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$.

Remark 3.5.11. First of all, in case $u = \chi_E$ with E a set of finite perimeter, J_u coincides with the reduced boundary $\partial^* E$ up to a set of $\mathcal{H}^{N-1}$-measure zero (see [8, Example 3.68]). At this point, it is also interesting to observe that the theorem we just stated actually generalizes Theorems 3.5.3 and 3.5.4 of the previous section.

Equality for purely jump BV functions

The following theorem was announced in the conference papers [100, 101].

Theorem 3.5.12. *Let Ω be an open set in $\mathbb{R}^N$ and $u \in BV(\Omega)$ whose diffuse part vanishes* i.e. *such that $Du = [u]\nu\mathcal{H}^{N-1}_{|J_u}$, then*

$$\tilde{J}(u) = J(u) = \int_\Omega |Du|.$$

Remark 3.5.13. In light of Remark 3.5.11, the previous theorem generalizes the one given for sets of finite perimeter. Nonetheless, this theorem is much more general since it proves that the equality remains valid even though the Hausdorff measure $\mathcal{H}^{N-1}(J_u)$ is not finite. Actually, only the rectifiability of J_u, and therefore the existence of tangents in a weak sense, really matters.

Proof. Step 1: First, we assume that $\mathcal{H}^{N-1}(J_u) < +\infty$. The strategy of this proof is similar to that of Theorem 3.5.5. Thus, since

$$\tilde{J}(u) = \sup_{\substack{\nabla w \in C_c(\Omega)^N \\ \|\nabla w\|_\infty \le 1}} \int_\Omega \nabla w \cdot Du,$$

we have to approximate $\int_\Omega |Du|$ by $\int_\Omega \nabla w_\varepsilon . D\chi_E$, as in the proof of Theorem 3.5.5. To begin with, let us localize the problem. First we need to fix some $0 < \varepsilon < 1$. Theorem 3.5.10 asserts that there exists $r_0 > 0$ such that for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$ and $r_x \geq 0$,

$$\int_{B(x,r_x)} |Du| = \omega_{N-1} {r_x}^{N-1} \left(1 + \eta_1(r_x)\right) \big|[u](x)\big|$$
$$\text{and} \int_{B(x,r_x)} \nu(x) \cdot Du = \omega_{N-1} {r_x}^{N-1} \left(1 + \eta_2(r_x)\right) [u](x),$$

with $|\eta_1(r_x)|, |\eta_2(r_x)| < \varepsilon$ whenever $r_x < r_0$.

The covering theorem asserts that the family $\left(B(x, r_x)\right)_{x,r_x}$ admits a countable and disjoint subcovering, denote in the sequel $(B_i = B(x_i, r_i))_{i\in\mathbb{N}}$, and whose balls satisfy

$$\sum_{i\in\mathbb{N}} \mathcal{H}^{N-1}(B_i) = \mathcal{H}^{N-1}(J_u).$$

We will not deal with the construction of the functions $w_{i,\varepsilon}$ in each ball and the problems that might occur at the boundary to define a global w_ε. We already took care of this in the proof of Theorem 3.5.5. In the sequel, it is enough to know that $w_\varepsilon = w_{i,\varepsilon}$ in each B_i and that $\nabla w_{i,\varepsilon} = \operatorname{sgn}([u](x_i))\nu_{x_i}$ in B_i.

Then, by the covering result, one gets

$$\int_\Omega \nabla w_\varepsilon \cdot Du = \sum_{i\in\mathbb{N}} \int_{B_i} \nabla w_{i,\varepsilon} \cdot Du.$$

Though, for any $i \in \mathbb{N}$,

$$\begin{aligned}
\int_{B_i} \nabla w_{i,\varepsilon} \cdot Du &= \omega_{N-1} r_i^{N-1} \left(1 + \eta_2(r_i)\right) \operatorname{sgn}([u](x_i))[u](x_i) \\
&= (1 + \eta_1(r_i))^{-1}(1 + \eta_2(r_i)) \int_{B(x_i,r_i)} |Du| \\
&\geq (1+\varepsilon)^{-1}(1-\varepsilon) \int_{B(x_i,r_i)} |Du|.
\end{aligned}$$

This is where we used several times Federer-Vol'pert's theorem. *In fine*, summing with respect to i, one has

$$\tilde{J}(u) \geq \limsup_{\varepsilon\to 0} \int_\Omega \nabla w_\varepsilon \cdot Du \geq \int_\Omega |Du|.$$

Step 2: In case $\mathcal{H}^{N-1}(J_u) = +\infty$, it is proven in [8, Proposition 4.2] that J_u is a Borel set that is σ-finite with respect to $\mathcal{H}^{N-1}$. This means that there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^N$ such that $J_u = \bigcup_{n\in\mathbb{N}} K_n$ and

$\mathcal{H}^{N-1}(K_n) < +\infty$. Then, by the previous reasoning, for fixed $n \in \mathbb{N}$ and $\varepsilon > 0$, one can construct a function $w_{\varepsilon,n}$ such that

$$\int_\Omega \nabla w_{\varepsilon,n} \cdot Du_{|K_n} \geq (1+\varepsilon)^{-1}(1-\varepsilon)\int_\Omega |Du|_{|K_n}$$

and the conclusion follows in the same way as in first step.

□

Remark 3.5.14. Our proof can be readily adapted to show that the so-called TGV^2 introduced in both [26, 40] equals the total variation for functions with vanishing diffuse part $Du = [u]\nu\mathcal{H}^{N-1}_{|J_u}$. In other words, one has

$$J(u) = \tilde{J}(u) = TGV^2(u) = TV(u).$$

Example 3.5.15. In some simple cases, it is actually possible to shorten the previous proof. Suppose, for instance, that the function $u \in BV(\Omega)$ is of the form $u = \sum_i \lambda_i \chi_{E_i}$ and the sets E_i form a countable increasing sequence, that is to say, one has $\bar{E}_i \subset\subset E_{i+1}$. One assumes in addition that each boundary ∂E_i is of class $C^{1,1}$. In this case, one can actually exhibit a function w that is smooth and such that

$$\tilde{J}(u) = \int_\Omega \nabla w \cdot Du = \int_\Omega |Du|.$$

Since $(E_i)_i$ is increasing, there exists $(V_i)_i$ a sequence of open and disjoint neighborhoods such that $\partial E_i \subset V_i$ and $\partial E_{i+1} \subset V_{i+1}$. As explained in Remark 3.3.5, it is not necessary to define w outside the disjoint union $\bigsqcup_i \partial V_i$, which is a neighborhood of $\operatorname{supp}(Du) = \bigsqcup_i \partial E_i$. Then let us set

$$w = -\sum_i \bar{d}(\cdot, E_i) \text{ in } V_i$$

where we recall that $\bar{d}$ is the signed distance defined by

$$\bar{d}(x, E_i) = d(x, \Omega \setminus E_i) - d(x, E_i).$$

Then, Theorem 3.5.8 asserts that w is $C^{1,1}$ in a neighborhood of $\operatorname{supp}(Du)$ and that $\nabla w = -\nu$. As a consequence,

$$\tilde{J}(u) \geq \int_\Omega \nabla w \cdot Du = -\sum_i \int_{\partial E_i} \nu \cdot Du = \int_\Omega |Du|$$

and the conclusion follows.

3.6 Application to image processing

3.6.1 ROF revisited

In 1992, Rudin, Osher and Fatemi (ROF) introduced the total variation, in their article [141], as a regularizing criterion for inverse problems in imaging. Total variation has been fruitful in image restoration since it can regularize images without smoothing the edges. We propose a possible approach to tackle the problem of restoring a degraded image by replacing the TV term by the functional $\tilde{J}$. Given $g \in L^2$, we are thus interested in the problem

$$\min_{u \in L^2(\Omega)} \mathcal{F}(u) = \lambda \tilde{J}(u) + \frac{1}{2}\|u - g\|_2^2, \tag{3.6.1}$$

parametrized by $\lambda \geq 0$. We recall that Corollary 3.3.6 asserts that $\tilde{J}(u)$ is well-defined for any $u \in L^2(\Omega)$.

Proposition 3.6.1. *Let $\Omega \subset \mathbb{R}^N$ be open and $g \in L^2(\Omega)$. Then $\mathcal{F}$ has a unique minimizer u_λ in $L^2(\Omega)$.*

Proof. Consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ such that $\mathcal{F}(u_n) \to \inf_\Omega \mathcal{F}$. As $\mathcal{F}(u_n) \leq \mathcal{F}(0) < +\infty$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Then up to extraction of a subsequence, still denoted $(u_n)_{n\in\mathbb{N}}$, it converges in $L^2(\Omega)$ weakly and also pointwise to some u_λ. Since $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence, we observe convergence of the norms. Thus $(u_n)_{n\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ and by lower-semicontinuity of $\tilde{J}$ (Corollary 3.3.6) together with Fatou, we get

$$\mathcal{F}(u) \leq \liminf_{n\to\infty} \mathcal{F}(u_n) = \inf_\Omega \mathcal{F}(u_\lambda).$$

This proves the existence of a minimizer, namely u_λ. It is unique by strict convexity of $\mathcal{F}$. $\square$

Remark 3.6.2. We did not need any compactness result for functional $\tilde{J}$. Only the L^2-semicontinuity is important.

Once, the existence of a minimizer is established, it is often interesting to compute an explicit solution. This is the object of the next part.

3.6.2 An explicit solution

In this section, let us assume that $N \geq 2$ and $\Omega = \mathbb{R}^N$.

Proposition 3.6.3. *Let $g = C\chi_{B(0,1)}$ and $\lambda \geq 0$. Then, if $C \geq \lambda N$, the minimizer of $\mathcal{F}$ is*

$$u_\lambda = (C - \lambda N)\chi_{B(0,1)}.$$

Remark 3.6.4. This is to be compared with the well-known result for TV (see [13]): under the previous assumptions, the minimizer of (ROF) is

$$u_\lambda = (C - \lambda N)\chi_{B(0,1)}.$$

Proof of Proposition 3.6.3. If $N = 2$, let us first set

$$w(x) = \lambda \times \begin{cases} -\frac{1}{2}|x|^2 & \text{if } |x| \leq 1, \\ -\ln|x| - \frac{1}{2} & \text{otherwise,} \end{cases}$$

and for $N \geq 3$,

$$w(x) = \lambda \times \begin{cases} -\frac{1}{2}|x|^2 & \text{if } |x| \leq 1, \\ \frac{1}{N-2}\frac{1}{|x|^{N-2}} + C' & \text{otherwise.} \end{cases}$$

Then

$$\nabla w(x) = \lambda \times \begin{cases} -x & \text{if } |x| \leq 1, \\ -\frac{x}{|x|^N} & \text{otherwise,} \end{cases}$$

hence

$$\Delta w(x) = \begin{cases} -\lambda N & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

This tells us that

$$u_\lambda - g = \Delta w. \tag{3.6.2}$$

Thus,

$$\int_\Omega \nabla w \cdot Du_\lambda = \int_\Omega |Du_\lambda|.$$

From Theorem 3.3.4 and the fact that $\tilde{J}(u) \leq \int_\Omega |Du|$, it follows

$$\lambda \tilde{J}(u_\lambda) = \int_\Omega \nabla w \cdot Du_\lambda = -\int_\Omega u_\lambda(u_\lambda - g)$$

Now considering some $v \in L^2(\Omega)$, (3.6.2) implies that

$$\lambda \tilde{J}(v) \geq \int_\Omega \nabla w \cdot Dv = -\int_\Omega v(u_\lambda - g).$$

Therefore,

$$
\begin{aligned}
\mathcal{F}(v)-\mathcal{F}(u_\lambda) &\geq -\int_\Omega (v-u_\lambda)(u_\lambda-g)+\frac{1}{2}\int_\Omega \left((v-g)^2-(u_\lambda-g)^2\right)\\
&= \int_\Omega (v-u_\lambda)(g-u_\lambda)+\int_\Omega (v-u_\lambda)\left(\frac{u_\lambda+v}{2}-g\right)\\
&= \frac{1}{2}\int_\Omega (u_\lambda-v)^2 \geq 0
\end{aligned}
$$

hence u_λ is the minimum of $\mathcal{F}$. □

3.6.3 Numerical aspects

Discretization

From now on, an image u will be represented by an $N\times N$ matrix with real entries *i.e.* an element of $X=\mathbb{R}^{N\times N}$. To simplify matters in the sequel, especially when we shall consider the discrete Fourier transform of u, we assume that the image u is also periodic and defined for all $k\in\mathbb{Z}$ by $u_{i+kN,j+kN}=u_{i,j}$ with $i,j\in\{1,...,N\}$.

To define the total variation of the image u, we first have to introduce a discretized version of the gradient. For $u\in X$, it is the vector ∇u of $Y=X\times X$ given by

$$
(\nabla u)_{i,j}=\begin{pmatrix} u_{i+1,j}-u_{i,j}\\ u_{i,j+1}-u_{i,j}\end{pmatrix},
$$

for $i,j=1,...,N$.

Let us also introduce two important operators: the divergence $\operatorname{div} p$ of an element $p\in Y$ and the Laplacian Δv of an image v. By analogy with the continuous setting, we want them to satisfy

$$
\langle \operatorname{div} p, u\rangle_X = -\langle p,\nabla u\rangle_Y \text{ and } \Delta v=\operatorname{div}\nabla v, \tag{3.6.3}
$$

for all $u\in X$.

The classical discrete approach

Given a noisy image g which has also been exposed to a linear perturbation A, the Rudin, Osher and Fatemi method suggests to minimize the following functional

$$
\mathcal{E}(u)=\frac{1}{2}\|Au-g\|_2^2+\lambda TV(u) \tag{3.6.4}
$$

to restore the image g. The positive parameter λ controls the regularization level. Here, $TV(u)$ is the most simple approximation of the total variation of $u\in X$

and is defined by

$$TV(u) = \sum_{i,j} |(\nabla u)_{i,j}|,$$

where $|\cdot|$ is simply the Euclidian norm of $\mathbb{R}^2$.

The discrete approach based on the variant $J(u)$

Let $u \in X$ be an image. The main idea is to replace the TV term in (3.6.4) by

$$J(u) = \min_{\substack{p \in Y \\ \Pi p = \nabla u}} \|p\|_1$$

where on the one hand, $\|p\|_1 = \sum_{i,j} \sqrt{\left(p^1_{i,j}\right)^2 + \left(p^2_{i,j}\right)^2}$ when $p = (p^1, p^2) \in X \times X$ and on the other hand, Π is the projection on the gradients defined by $\Pi p = \nabla \bar{v}$, where $\bar{v}$ realizes the minimum

$$\min_{v \in X} \|\nabla v - p\|_2. \tag{3.6.5}$$

Here $\|\cdot\|_2$ is the Euclidian norm of Y. Remark by the way that we have

$$J(u) \leq TV(u)$$

for any $u \in X$. This is a straightforward consequence of the definition.

Now, it should be noted that, the solution of (3.6.5) is characterized by the Euler-Lagrange equation

$$\nabla^*(\nabla u - p) = 0$$

or, using the notation introduced in (3.6.3),

$$\Delta u = \operatorname{div} p,$$

(we recall that our operators ∇, div and Δ are here discrete operators with periodic boundary conditions). Therefore,

$$J(u) = \min_{\substack{p \in Y \\ \operatorname{div} p = \Delta u}} \|p\|_1.$$

Hence, the Rudin, Osher and Fatemi's problem expressed in terms of this new functional consists in minimizing

$$\mathcal{F}(p) = \frac{1}{2}\|Au - g\|_2^2 + \lambda \|p\|_1$$

over (p, u) which satisfy the constraint $u = \Delta^{-1} \operatorname{div} p$.

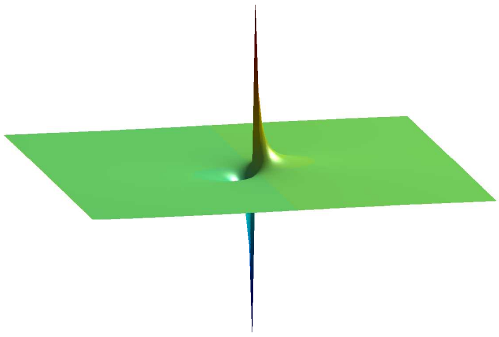


Figure 3.4: Approximation of a basis element $\Delta^{-1}\operatorname{div}(\delta_0, 0)$ of J

Lately, minimization of such functionals has attracted lot of attention and was the subject of many papers. Among those, the recent articles [132, 21, 59] (see Chapter 5 for further details) focus on the minimization of objective functions which can be decomposed as a sum namely

$$\min_x F(x) + G(x)$$

where G is a continuously differentiable convex function whose gradient is Lipschitz continuous and F is a continuous convex function which is possibly nonsmooth but is simple in the sense that its proximal operator is easy to compute (see Combettes and Wajs [69] for more on the subject). These characteristics suit perfectly the two terms composing $\mathcal{F}$ and we henceforth denote

$$G(p) = \frac{1}{2}\|A\Delta^{-1}\operatorname{div} p - g\|_2^2 \text{ and } F(p) = \|p\|_1.$$

In the sequel, we detail, as an example, the implementation of the algorithm of Beck-Teboulle. As just said, we could have also used the Primal-Dual algorithm instead. Now let us denote L the Lipschitz constant of ∇G. In their article, Beck and Teboulle describe the following scheme to construct a minimizing sequence $(p_n)_{n\in\mathbb{N}}$ for $\mathcal{F}$:

Algorithm 3.1 proposed by Beck and Teboulle in [21]

- **Initialization:** $q_1 = p_0 \in Y,\ t_1 = 1,$
- **Iterations:** For $n \geq 1$ consider the following updates

$$p_n = \operatorname{argmin}\left\{G(q_n) + \langle p - q_n, \nabla G(q_n)\rangle_Y + \frac{L}{2}\|p - q_n\|_2^2 + \|p\|_1,\ p \in Y\right\},$$

$$t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2},$$

$$q_{n+1} = p_n + \left(\frac{t_n - 1}{t_{n+1}}\right)(p_n - p_{n-1}).$$

In the present case, one actually has

$$p_n = \operatorname{argmin}\left\{ \sum_{i,j} \left(p_{i,j} \cdot (\gamma_n)_{i,j} + \frac{L}{2}|p_{i,j} - (q_n)_{i,j}|^2 + \lambda |p_{i,j}| \right), p \in Y \right\},$$

where $\gamma_n = \nabla G(q^n) = (A\Delta^{-1} \operatorname{div})^*(A\Delta^{-1} \operatorname{div} q_n - g)$, that can be computed easily with the Fourier transform if A itself can be calculated thanks to the Fourier transform (this is true for the convolution or the Radon transform for instance). The previous sum can be minimized termwise, thus it follows

$$(p_n)_{i,j} = \max\left(0, \left|(x_n)_{i,j}\right| - \frac{\lambda}{L}\right) \frac{(x_n)_{i,j}}{\left|(x_n)_{i,j}\right|}$$

with $(x_n)_{i,j} = (q_n)_{i,j} - \frac{(\gamma_n)_{i,j}}{L}$. Moreover, in case A is the convolution with a Gaussian, one has $L = \frac{1}{2}(1 - \cos(\frac{2\pi}{N}))^{-1}$.

Let us remark that in the algorithm proposed by Beck and Teboulle, each iteration requires only one computation of the gradient ∇G. Whereas, a similar algorithm proposed by Nesterov in [132], each iteration demands the computation of the gradient ∇G at two different locations, which slows down notably the overall calculation. Though, from a theoretical point of view, the algorithms of Nesterov and Beck-Teboulle both converge as $O(\frac{1}{n^2})$ (see Chapter 5).

Experimental results

Characteristic function of a square

Even though we proved that J, $\tilde{J}$ and TV coincide on characteristic functions of sets of finite perimeter, it is not true that the minimizers of (3.6.1) are identical. In the following simulation, the datum is the characteristic of a square S:

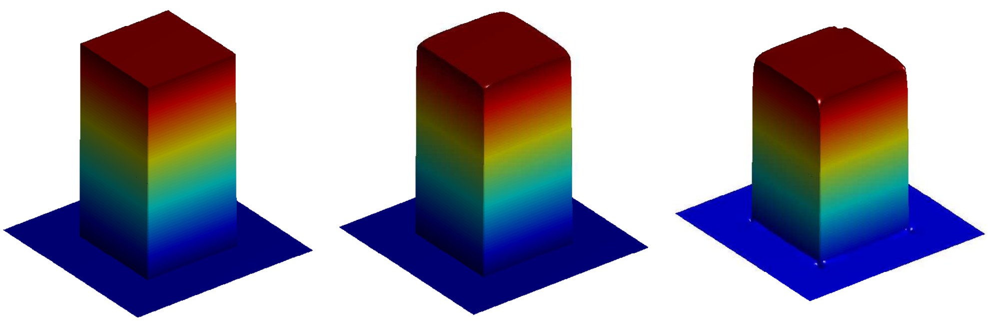


Figure 3.5: Graph of $g = \chi_S$, the TV-minimizer and the J-minimizer, respectively. The regularization parameter is kept constant.

Absence of staircasing

Let us consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = x^2 - y^2$. As seen in Proposition 3.3.1, $\tilde{J}(g) = 0$ which means that it should be restored perfectly whereas the minimizer of TV will necessary have a flat region in a neighborhood of the origin. This can actually be seen in the following experiments:

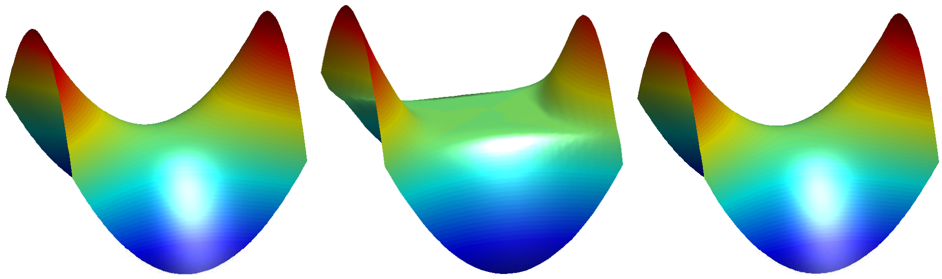


Figure 3.6: Graph of g, the TV-minimizer and the $\tilde{J}$-minimizer, respectively. One kept $\lambda = 100$.

Real images

In the sequel, we compare J with TV for natural images. The images we considered are not periodic so we used the periodic+smooth decomposition introduced by Moisan in [125] to get a proper periodic image. This way we can use the discrete Fourier transform without having any artifact near the boundary. For the TV-minimization we use the Primal-Dual algorithm whereas we get the J-restored images with the algorithm of Beck-Teboulle. The resulting images are obtained after 250 iterations of both of these algorithms, which is fair enough according to the results of Section 5.6.

Here follows a first experiment for a clean image:

Figure 3.7: Image *Lena*, its TV-minimizer and its J-minimizer obtained with $\lambda = 25$, respectively.

Let us remark that the underlying vectorfield p which minimizes $\mathcal{F}$ is quite different from the gradient of the J-restored image though p characterizes quite well the boundaries in the image:

Figure 3.8: Vectorfield p minimizing $\mathcal{F}$ and $\nabla u = \nabla \Delta^{-1} \operatorname{div} p$, respectively. Hue indicates the field direction, and saturation indicates the magnitude.

If one increases the value of the parameter λ one expects the same smoothing effect as TV. Though, as for TV the edges are quite well preserved:

Figure 3.9: *Lena*, its TV-minimizer and its J-minimizer for $\lambda = 100$, respectively.

Noisy image 1:

PSNR=22.1 dB PSNR=29.4 dB PSNR=29.3 dB

Figure 3.10: Image *Boats* that underwent an addition of a white Gaussian noise of standard deviation $\sigma = 20$, its TV-minimizer and its J-minimizer obtained with $\lambda = 25$, respectively.

Noisy image 2:

Figure 3.11: Image *Barbara* that underwent an addition of a white Gaussian noise of standard deviation $\sigma = 20$, its TV-minimizer and its J-minimizer obtained with $\lambda = 25$, respectively.

Blurry and noisy image:

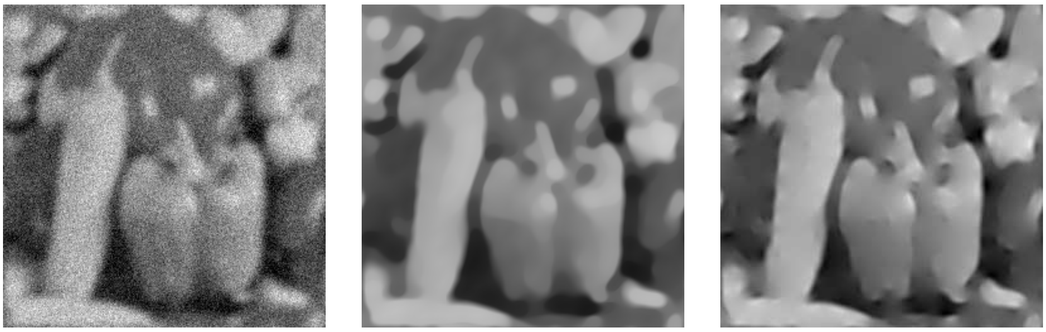


Figure 3.12: Image *Peppers* that underwent a Gaussian blur of standard deviation $\sigma = 3$ and was contaminated by a white Gaussian noise of standard deviation $\sigma = 20$, the TV-minimizer and the J-minimizer obtained with $\lambda = 25$, respectively.

3.7 Conclusion and perspective

In this chapter, we studied an alternative for the total variation and we gave some applications to image processing. The actual functional we propose coincides with the total variation on cartoon images. This was proven by first establishing a dual formulation and then using some known results on the regularity of the discontinuity set of a function of bounded variation.

The latter formulation also helps proving a first result toward a Poincaré-type inequality. It is actually not clear to us whether such an inequality can hold in this context.

Another interesting question would be to prove that the canonical space, normed by the functional we proposed, is embedded in some Besov space. By definition, this canonical space contains BV.

The "projection on the gradients" of the introduction can be rewritten in $\mathbb{R}^N$ in terms of the Riesz transformation (see [146] on the subject). This seems to be related to the work of Michaël Unser [151] where steerable wavelets are defined via Riesz tranformations. This connection could be investigated in a future work.

Another related question that remains unsettled: does the Helmholtz decomposition make sense for measures? The answer is not clear though in [19], the authors prove that such a result does not hold for 1-forms.

CHAPTER 4

Adapted Basis for Non-Local Reconstruction of Spectrum

Contents

4.1 Introduction

Traditional methods in image processing are based on local properties of images (wavelets, total variation). Recently, state of the art results were achieved by non-local or semi-local methods that exploit redundancies in images. The idea of using self-similarities in images was first exploited in [44] for the denoising problem. They indeed propose a (semi-)non-local filter that averages similar patterns of a noisy image $g = g_0 + n$ defined on $\Omega \subset \mathbb{R}^2$ even though these self-similarities occur at large distance. The filter reads as follows

$$\text{NLMeans}(g)(x) = \frac{1}{C(x)} \int_\Omega g(y) \exp\left(-\frac{\|p_g(x) - p_g(y)\|}{h}\right) dy$$

where $p_g(x)$ and $p_g(y)$ are patches of the noisy image g and h determines the selectivity of the similarity measure. As already noted the idea of looking at such a term dates back to the filter

$$Y(g)(x) = \frac{1}{C(x)} \int_\Omega g(y) \exp\left(-\frac{\|g(x) - g(y)\|}{h}\right) dy$$

proposed in [157] which merely compares gray levels. It was actually observed that the NLMeans filter yields much better results if one seeks for similar patches in a subdomain of the image, referred to as the *window*. This is why this filter is not fully non-local. Many modifications of this filter were considered for denoising purposes (adaptive h, adaptive window [104], non-square patches [73], shape adaptive patches [70]).

Though, an important drawback of the NLMeans filter is that the image is not processed if no similar patches are found. This phenomenon is referred to as the *rare patch effect*. To remedy this artifact, the authors of [110] propose to mix local and non-local methods by preprocessing the patches with the total variation. The regularization parameter being set locally so that one gets enough similar patches.

The use of patches has been widely adopted in the image processing community and in the recent years, these non-local methods were extended to the study of general inverse problems (see [91, 90, 135, 15]). Such a general inverse problem can be modelled as follows:

$$g = Ag_0 + n$$

where g_0 is the original image defined on $\Omega \subset \mathbb{R}^2$, A is a linear transformation and n is a Gaussian white noise. The classical local methods were adapted by considering a *pointwise Non-Local regularization* as introduced in [91]

$$J_w(u) = \int_{\Omega\times\Omega} |u(x) - u(y)|^\alpha w(x, y)\eta(x - y)dxdy$$

for some $\alpha \geq 1$ and where $w(x, y) = \exp\left(-\frac{\|p_g(x)-p_g(y)\|}{h}\right)$ is the weight function used in the NLMeans filter and is based on the Sum of Squared Differences (henceforth called SSD distance). As for η, it is a function of compact support centered at the origin. It indicates that patches that are too far from each other should not be taken into account. The support of η is referred to as the *search window*.

Peyré et al. proposed in [136] (see also [15]) to replace the latter term by a *patchwise Non-Local regularization*

$$J_w(u) = \int_{\Omega\times\Omega} \|p_u(x) - p_u(y)\| w(x, y)\eta(x - y)dxdy$$

that is able to reconstruct entire patches. To get a restored image one therefore has to minimize the following non-local energy

$$\mathcal{E}(u) = \frac{1}{2}\|Au - g\|_2^2 + \lambda J_w(u).$$

It is further remarked that one can enhance the restoration by recomputing the weight $w(x, y)$ on the being processed image. In [15, 136], the authors proposed a general framework where the weight $w(x, \cdot)$ is interpreted up to renormalization as a density of probability and is an unknown of the problem. The energy to be minimized is then

$$\mathcal{E}(u, w) = \frac{1}{2}\|Au - g\|_2^2 + \lambda J_w(u) - h^2 \int_{\Omega\times\Omega} w \log w \qquad (4.1.1)$$

which involves the potential energy of $w(x, \cdot)$, used to infer unknown probability distributions. The energy $\mathcal{E}$ is obviously not convex in w and an alternate coordinate descent gives back the SSD-based weight but computed this time on the being processed u. This corresponds actually to the aforementioned weight recomputation procedure.

In this chapter, we are going to deal with the problem of restoring missing Fourier coefficients of an image. Our aim is to adapt the abovementioned patchwise Non-Local Regularization by proposing an original way of computing the distance between patches, that is adapted to the problem of spectrum completion. This problem, though interesting by itself, has several applications for general inverse problems. This is the object of the next part.

4.2 Motivation

In this section, we consider the problem of retrieving missing Fourier coefficients of a raw data. This problem has important applications since several important inverse problems can be casted into this framework:

- The tomography problem for medical imaging or seismic imaging. In this case Fourier coefficients usually lie on straight lines that are either parallel or that cross at the origin.
- Aperture Synthesis for spatial imaging where the corruption is given by the mask considered in Figure 4.5 (see [102] for instance for further details).
- The zooming problem: this is of some importance nowadays for the transition of SD (Standard Definition) videos to HD (High Definition) ones.
- The inverse scattering problem where one is interested in recovering the shape of an object that is hidden in a medium using electromagnetic or acoustic waves.

As an exemple, we decide to detail the latter application. This topic is the object of two recent surveys [68, 67] and the text book [129]. Scattering theory is concerned with the effects of static objects on traveling waves. Let us consider an *incident acoustic plane wave* $u^i(x) = e^{ikx\cdot d}$ propagating in the direction of the unit vector d in an isotropic medium. Here $k > 0$ is the *wave number*. In case there is an inhomogeneity D (one should think of a hidden object in the medium), the wave will be "scattered" and give rise to another wave u^s. The resulting wave $u = u^i + u^s$ satisfies in $\mathbb{R}^N$ the Helmholtz equation

$$\Delta u + k^2(1 - \chi_D)u = 0$$

and the *scattered wave* u^s satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to+\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0,$$

where $r = |x|$ and the limit is uniform in the direction $\hat{x} = x/|x|$. Then, it is known that u^s has an asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|}u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|^2}\right),$$

where $u_\infty(\hat{x}, d)$ is known as the *far field pattern* and gives an idea of the behavior of the scattered wave at large distance. Here, $\hat{x}$ is the observation direction and d is the incident wave direction. The *direct problem* amounts to find $u_\infty(\hat{x}, d)$ whereas the *nonlinear inverse problem* takes the direct method as a starting point and asks what is the nature of the scatterer D that gave rise to such a farfield. This problem is well-posed since the knowledge of $u_\infty(\hat{x}, d)$ determines uniquely the obstacle D. A significant development in this field is the introduction of the *factorization method* that is detailed in the references above (see also [105]). In what follows we assume that k is sufficiently small so that the *Born approximation* implies that

$$u_\infty(\hat{x}, d) \approx \int_{\mathbb{R}^N} \chi_D(y) e^{-ik(\hat{x}-d)\cdot y} dy. \tag{4.2.1}$$

This means that $u_\infty(\hat{x}, d)$ can be interpreted as Fourier coefficients in the ball of radius $2k$. In practice, one has only limited incident waves $(d_j)_j$ and limited measurements $(\hat{x}_i)_i$ corresponding to the values $u_\infty(\hat{x}_i, d_j)$. We thus have a sampling of the Fourier coefficients of $\chi_D(y)$ as one can see in Figure 4.1.

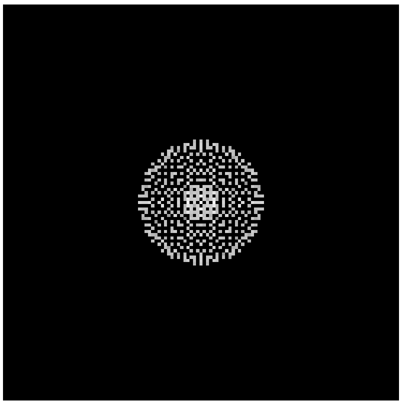


Figure 4.1: Spectrum obtained with 32 incident planewaves and 32 measurement directions.

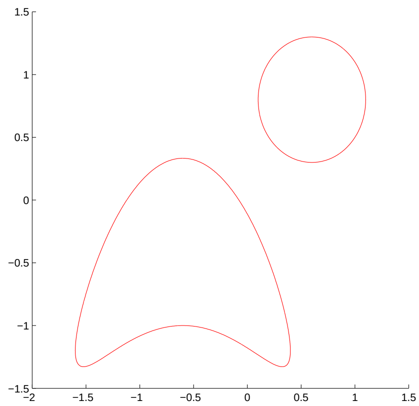


Figure 4.2: Scatterer D.

Irregular sampling of the Fourier coefficients is a difficult problem (see for instance [121]), though in the present case this is not really an issue. Indeed, assuming that D is bounded its Fourier transform is actually a C^∞ function and can therefore be interpolated on a uniform grid. This allows us to use the `fft`. This is sometimes referred to as the *regridding* procedure.

In [96], the author uses the total variation for interpolating missing parts of the spectrum. The problem has also been tackled in [124] where a weighted total variation was considered. Let us therefore compare the performance of this approach with that of the factorization method thanks to a numerical test. We consider the object that is bounded by the red curves depicted in Figure 4.2. The inverse problem is then solved given 32 incident waves and 32 measurement directions*. The wave number is $k = 3\pi$. The factorization method, which

*I would like to thank Armin Lechleiter who provided me with the solution of the direct problem and the code for the factorization method.

performs quite well for small values of k, yields quite poor results in this extreme case:

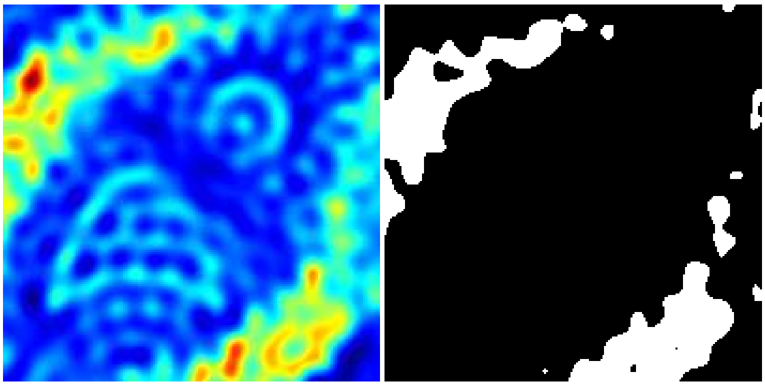


Figure 4.3: Solution obtained by the classical *factorization method* (left) [105] and then thresholded (right).

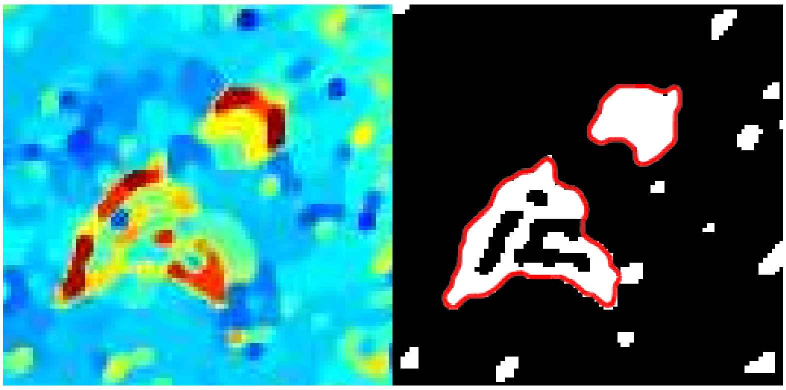


Figure 4.4: Solution obtained by TV minimization (left), then thresholded and segmented (red curve on the right). For details on segmentation see [64].

The total variation method performs very well in this case. Indeed, from partial and noisy measurements, we were able to distinguish two objects that are merely separated by $3/4\lambda$ where $\lambda = 2\pi/k$ is the *wavelength*. This value is close to the theoretical diffraction limit that is $\lambda/2$. Moreover, we would like to point out that it is not necessary to tune a Lagrange multiplier λ. Indeed if one denotes M the discrete set of points where u_∞ is known one can simply minimize the constrained problem

$$\min_{\mathcal{F}u_{|M}=u_\infty} TV(u),$$

to get the restoration shown above. Total variation was already used for solving the nonlinear inverse scattering problem in [152]. The idea of considering the inverse problem within the Born approximation as an inpainting problem in the Fourier domain is discussed in [77].

For this family of problems, can we do better than using TV? The object of this work is to propose an adequate regularization for the problem of interpolating missing Fourier coefficients. As said in the introduction, some recent papers address the problem of solving general inverse problems in imaging by means of non-local methods. These methods obviously apply to the problem of retrieving missing Fourier coefficients. We should also mention the not so different problem of restoring missing wavelet coefficients which is treated in [63, 158] by classical local and then non-local methods.

In these approaches, a critical step is to compute the distance between patches of the corrupted image. Usually this similarity measure is simply taken to be

$$\delta(p_g(x_k), p_g(x_\ell)) = \|p_g(x_k) - p_g(x_\ell)\|_{\ell^2}$$

where $p_g(x_k), p_g(x_\ell)$ are two patches of the corrupted image g. The idea of considering the SSD distance dates back to the NLMeans [44] and to our knowledge no other distance has been considered since then.

In this work, we propose a variant of these recent non local methods by designing an original patch-wise distance that is adapted to the problem of spectrum completion. Our idea is to define atoms similar to Gabor atoms to test whether two regions of the corrupted image are similar. These atoms should not depend on the corruption process. This way two patches that are close in the clean image will be close in the corrupted image.

4.3 A Non-Local energy for the problem

Unless otherwise stated we place ourselves in $\mathbb{R}^2$ which will be equipped with the infinity norm still denoted $|\cdot|$. Considering squares instead of balls can be handy in image processing.

As seen in the previous section, the problem of reconstructing unknown coefficients of a Fourier series has several applications. Let us call $M \subset \mathbb{Z}^2$ the finite mask of points where the Fourier coefficients are known. Assuming that the image is periodic and defined on $\mathbb{T} = [0,1]^2$, one thus considers a clean image

$$g_0(x) = \sum_{k \in \mathbb{Z}^2} c_k e^{-2i\pi k \cdot x}$$

and a corrupted one

$$g(x) = \sum_{k \in M} c_k e^{-2i\pi k \cdot x}.$$

In the sequel we are going to consider the following generic mask M:

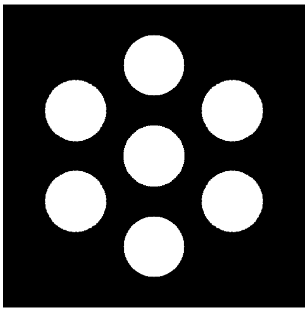


Figure 4.5: Corruption mask M.

We keep the Fourier coefficients corresponding to the white discs of radius r_M: this means that we capture low frequencies and some middle range frequencies but we discard all the high frequencies.

Let $\mathcal{M} = \{\sum_{k\in M} c_k e^{-2i\pi k\cdot x}, c_k \in \mathbb{R}\}$ and its complement $\mathcal{M}^{\perp} = \{\sum_{k\notin M} c_k e^{-2i\pi k\cdot x}, c_k \in \mathbb{R}$ and $\sum_{k\in\mathbb{Z}^2} |c_k|^2 < +\infty\}$. Let us also denote $P_{\mathcal{M}}$ the orthogonal projection on $\mathcal{M}$ and let us recall that the Fourier transform on the torus $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}^2)$ is an isometry and thus invertible. We therefore have a corrupted data $g \in \mathcal{M}$, obtained by

$$g = \mathcal{F}^{-1} \circ P_{\mathcal{M}} \circ \mathcal{F}(g_0) = \mathcal{F}^{-1}(\chi_M \mathcal{F}(g_0)),$$

that one has to interpolate. The corruption process results into a perturbation that is similar in nature to the Gibbs phenomenon. Note by the way that g is real-valued as soon as M is symmetric with respect to the origin.

In the sequel we shall call *patch* centered at $x_k \in \mathbb{R}^2$, and denote $p_g(x_k)$, the cropped image $g(\cdot - x_k)\varphi$ where φ is some mollifier function with support $B(0, \frac{\rho}{2})$ where ρ is the *patch size*.

Let us now assume that the original image is such that there are two distinct $x_k, x_\ell \in \mathbb{R}^2$ with

$$g_0(x - x_k)\varphi(x) = g_0(x - x_\ell)\varphi(x), \ \forall x \in \mathbb{R}^2.$$

This is to say that the two patches are similar in the original image. Then clearly one expects that the missing spectrum v solves

$$\min_{v\in\mathcal{M}^{\perp}} \int_{\mathbb{T}} \psi(x)^2 |(g+v)(x - x_k) - (g+v)(x - x_\ell)|^2 dx \qquad (4.3.1)$$

for some smooth ψ such that $\mathrm{supp}(\psi) \subset \mathrm{supp}(\varphi) = B(0, \frac{\rho}{2})$. This means that knowing that $p_{g_0}(x_k)$ and $p_{g_0}(x_\ell)$ are similar one can hope to get a restored spectrum by minimizing (4.3.1). The reconstruction is obviously not unique since modifying v out of $(x_k + \mathrm{supp}(\psi)) \cup (x_\ell + \mathrm{supp}(\psi))$ does not change the energy.

Numerical simulations show that the reconstructed v has the same support as $(x_k + \mathrm{supp}(\psi)) \cup (x_\ell + \mathrm{supp}(\psi))$ that is to say we have to take into account all the patches in the image to get a global reconstruction: we are therefore led to consider the following problem

$$\min_{v \in \mathcal{M}^\perp} \int_{\mathbb{T}} \sum_{(k,\ell)\in I} \psi^2 |(g+v)(\cdot - x_k) - (g+v)(\cdot - x_\ell)|^2 w(x_k, x_\ell) \tag{4.3.2}$$

where $w(x_k, x_\ell) = \exp\left(-\frac{\delta(x_k,x_\ell)}{h}\right)$ and $\delta(x_k, x_l)$, that is going to be defined in the sequel, should tell us from the corrupted image g whether one had for the original image $p_{g_0}(x_k) \sim p_{g_0}(\ell)$. This is going to be a critical step as can be seen in the following simulation where we consider the *oracle distance*

$$\delta(x_k, x_\ell) = \|p_{g_0}(x_k) - p_{g_0}(x_\ell)\|_2$$

computed on the clean image:

Original image	Corrupted image PSNR=20,0dB	Ideal weight PSNR=26.5dB	TV restored PSNR=21.0dB

Original image	Corrupted image PSNR=24,1dB	Ideal weight PSNR=29.8dB	TV restored PSNR=24.5dB

Figure 4.6: Non-Local restoration thanks to the oracle distance.

This oracle distance which obviously cannot be used in practice, leads to a very good restoration and shows that this model performs very well if one is able to define a good measure of similarity for patches.

4.4 Construction of Atoms adapted to the corruption

Our aim is to define a notion of (semi-)distance that is not modified through the corruption process $\mathcal{F}^{-1}(\chi_M \mathcal{F}(\cdot))$. This way, if two points are close in the original image with respect to this distance they will remain close in the perturbed image.

To do so, our idea is to find a family of test functions $(\phi_\alpha)_\alpha$ dense in $\mathcal{M}$, that do not depend on g and such that

$$g * \phi_\alpha = g_0 * \phi_\alpha, \ \forall \alpha.$$

These atoms $(\phi_\alpha)_\alpha$ will obviously depend on the mask M. In particular, one should have for any $g \in \mathcal{M}$,

$$\begin{aligned} \langle g, \phi_\alpha \rangle &= \langle \mathcal{F}^{-1}(\chi_M \mathcal{F}(g)), \phi_\alpha \rangle \\ &= \langle g, \mathcal{F}^{-1}(\chi_M \mathcal{F}(\phi_\alpha)) \rangle \end{aligned}$$

which means that $\mathcal{F}\phi_\alpha = \chi_M \mathcal{F}\phi_\alpha$ and therefore $\mathrm{supp}(\mathcal{F}\phi_\alpha) \subset M$ for any α. From the uncertainty principle (see [113] for instance), one knows that ϕ_α is not of compact support but one could still expect it to be localized possible in space. This can be ensured by defining these functions as minimizers of the following parametrized problems

$$\phi_\alpha = \operatorname{argmin} \left\{ \int_\Omega |\phi(x)|^2 |x|_2^p dx, \|\phi\|_2 \geq 1, \phi \in \mathcal{M}, \phi \perp \mathrm{Span}\{\phi_{\alpha'}, \alpha' < \alpha\} \right\}$$

where the moment $p > 1$.

In the end, we define for each ϕ_α

$$\begin{aligned} \delta_\alpha(x_k, x_\ell) &= |g * \phi_\alpha(x_k) - g * \phi_\alpha(x_\ell)| \\ &= |g_0 * \phi_\alpha(x_k) - g_0 * \phi_\alpha(x_\ell)|. \end{aligned}$$

We can then consider a measure of similarity between patches that is of the form

$$\delta(x_k, x_l) = \left(\sum_{\alpha < \alpha_0} \delta_\alpha(x_k, x_\ell)^2 \right)^{\frac{1}{2}},$$

where α_0 is a threshold that sets how localized the considered atoms ϕ_α are. The semi-distance δ is by definition invariant under the corruption process $\mathcal{F}^{-1}(\chi_M \mathcal{F}(\cdot))$.

In the discrete setting, the atoms introduced above can be computed iteratively by a projected gradient algorithm [130]. Then, if we consider the generic mask M introduced in the previous section we get the following atoms:

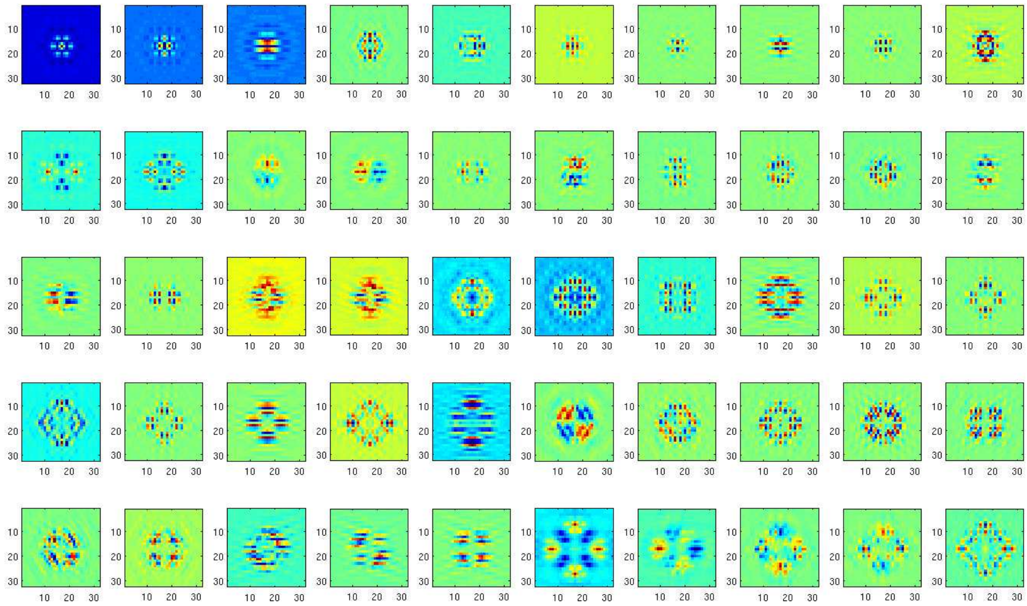


Figure 4.7: The 50 first atoms (zoomed in) with $p = 4$ adapted to the 128×128 mask M.

It is interesting to see that these atoms are really localized and can actually be used as test functions. Let us have a better look at the normalized 7 first atoms with their respective spectra:

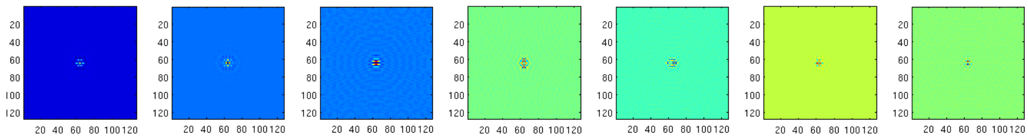


Figure 4.8: Atoms ϕ_n, $n = 1, ..., 7$ with $p = 4$ adapted to the 128×128 mask M,

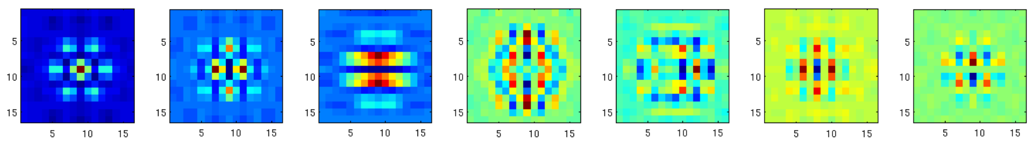


Figure 4.9: Atoms ϕ_n, $n = 1, ..., 7$, zoomed in,

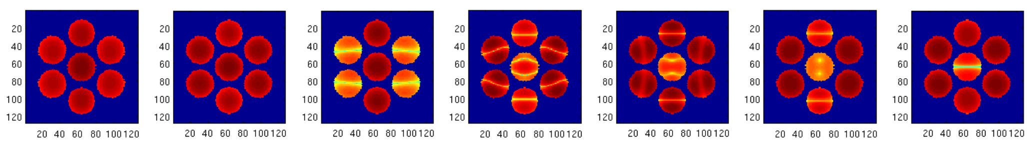


Figure 4.10: $\log(|\mathcal{F}(\phi_n)|)$, $n = 1, ..., 7$.

The atoms adapted to the mask M may recall the reader of the Gabor atoms (see [113]). However they have the advantage of having a prescribed spectrum and being as localized as possible. They are by definition the optimal functions satisfying these two conditions.

Let us now consider the following synthetic clean image

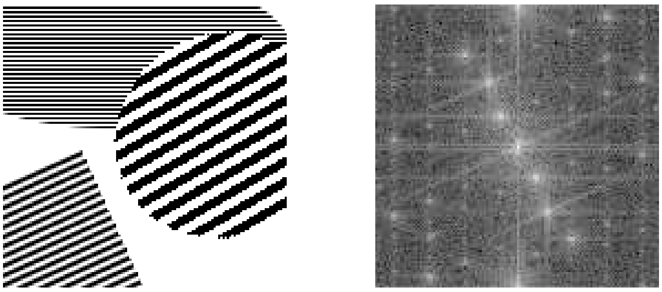


Figure 4.11: Image g_0 and its spectrum $\log|\mathcal{F}(g_0)|$,

and its corrupted version where we got rid of the high frequencies

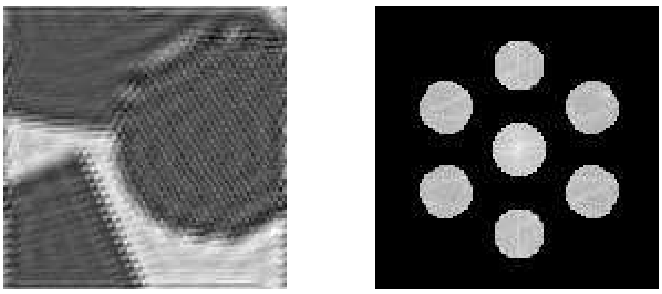


Figure 4.12: Image g and its spectrum $\log|\mathcal{F}(g)|$.

Let us now have a closer look at this image filtered by the atoms ϕ_n:

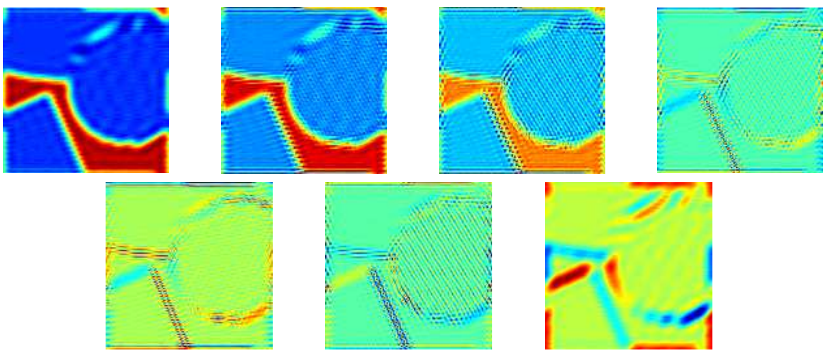


Figure 4.13: Filtered $g * \phi_n$, $n = 1, ..., 7$.

We can see that these atoms behave as Gabor atoms by capturing different patterns in the corrupted image. The different regions of g can thus be separated by analyzing the filtered $g * \phi_n$.

The previous atoms were computed for the generic mask considered in Figure 4.5. Though, our approach is quite general and can be adapted to any other corruption mask. As an example, here follow the atoms we get if we consider the mask that comes into play in the scattering problem:

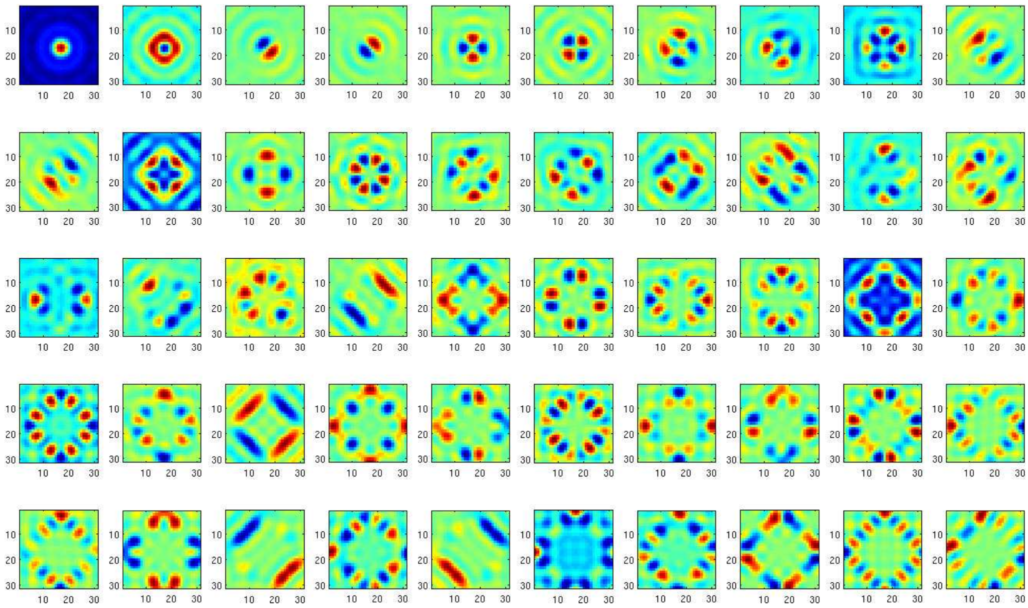


Figure 4.14: The 50 first atoms (zoomed in) with $p = 4$ adapted to the mask associated to the scattering problem.

In the following we depict the very first atoms with their spectra:

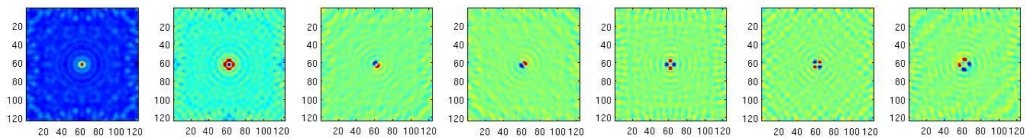


Figure 4.15: Atoms ϕ_n, $n = 1, ..., 7$ with $p = 4$ adapted to the scattering problem,

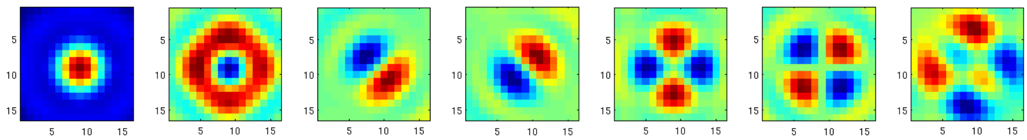


Figure 4.16: Atoms ϕ_n, $n = 1, ..., 7$, zoomed in,

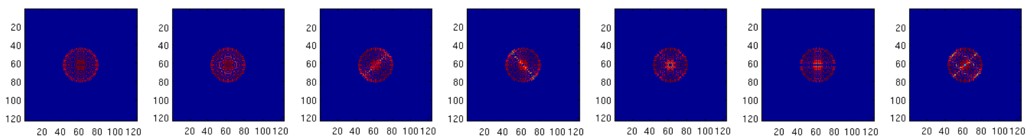


Figure 4.17: $\log(|\mathcal{F}(\phi_n)|)$, $n = 1, ..., 7$.

4.5 Distance map comparison

We are going to compare the performance of different similarity measures on patches. Let us consider the following image g_0 and its corresponding g:

Figure 4.18: Psychedelic *Lena* image g_0 that and the corrupted g obtained with the mask of Figure 4.5.

We fix one patch of the *corrupted image* g (which is indicated in green) and identify the 13 best matches (in red) according to

1. The atom-based ℓ^2 distance $\delta^1(x_k, x_\ell) = \left(\sum_{n=1}^{7} \delta_n(x_k, x_\ell)^2\right)^{\frac{1}{2}}$. We could have used a general ℓ^q, $q > 1$, but this choice does not seem to modify significantly the final results.

2. The ideal ℓ^2 distance $\delta^2(x_k, x_\ell) = \|p_{g_0}(x_k) - p_{g_0}(x_\ell)\|_2$ computed on the clean image g_0.

3. The SSD ℓ^2 distance $\delta^3(x_k, x_\ell) = \|p_g(x_k) - p_g(x_\ell)\|_2$ computed on the corrupted image g.

Example 1:

Figure 4.19: Best matches for the atom-based δ^1,

Figure 4.20: Best matches for the ideal distance δ^2,

Figure 4.21: Best matches for the SSD δ^3.

Example 2:

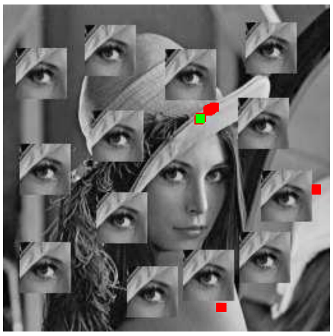


Figure 4.22: Best matches for the atom-based δ^1,

Figure 4.23: Best matches for the ideal distance δ^2,

Figure 4.24: Best matches for the SSD δ^3.

As can be seen in these two examples, the atom-based distance designed in the previous section seems to perform well and even better than the SSD distance in some cases.

4.6 Numerical experiment

From now on, we are going to make important assumptions that let us drastically improve the complexity of the minimization problem (4.3.2) and save memory:

(i) Two patches are unlikely to be the same if they are far from each other thus we can assume that for any fixed x_k, x_ℓ is a candidate if $|x_k - x_\ell| \leq \eta$ for some fixed η. This defines a neighborhood of candidates (also called *window*) centered at x_k.

(ii) If we consider two distinct points x_k, x_ℓ, there is some numerical evidence that minimizing (4.3.1) yields a $v_{k,\ell}$ whose support is actually $(x_k + \mathrm{supp}(\psi)) \cup (x_\ell + \mathrm{supp}(\psi))$. Therefore assuming that $\varepsilon \leq |x_k - x_\ell|$ for some positive $\varepsilon \leq \rho$ which is the patch size we can still get a global minimizer for (4.3.2). Although, usually $\varepsilon = 1$ pixel, assuming that $\varepsilon = 7$ pixels for a patch size $\rho = 7$ or 9 pixels (to let the reconstructed patches to overlap) results in an acceleration of order 10.

(iii) Once the weight w is computed, we can for fixed x_k keep only the m_0 best matches for x_l (see the previous examples).

Weight computation: Notice that for a fixed mask M, the atoms $(\phi_n)_{n=1,\dots,n_0}$ can be computed in advance and stored. Then $g_n = \phi_n * g$ can be readily, computed once for all, using the `fft`. Finally, for two fixed pixels x_k and x_ℓ

$$(\delta^1(x_k, x_l))^2 = \sum_{n=1}^{n_0} |g_n(x_k) - g_n(x_\ell)|^2$$

whereas

$$(\delta^3(x_k, x_l))^2 = \sum_{i=1}^{\rho^2} |p_g(x_k)(i) - p_g(x_\ell)(i)|^2.$$

Typically $\rho = 7$ or 9 pixels is the patch size. This suggests that the computation of δ^1 is faster as far as $n_0 \leq \rho^2$ which will actually be the case. In practice, this can result in an acceleration of order 10.

Energy minimization: The constrained minimization problem (4.3.2) involves a quadratic energy that can be minimized thanks to the conjugate gradient or also by Nesterov's algorithm [131]. We refer to Chapter 5 for further details on these algorithms. Notice that the assumptions $(i) - (iii)$ are important at this stage to accelerate the computation of the gradient of the energy.

Numerical results: In the following tests we consider three different energies:
- The constrained total variation: we get the restored spectrum $\mathcal{F}v$ by solving $\min_{v \in \mathcal{M}^\perp} TV(g+v)$.
- The constrained minimization problem (4.3.2) where we consider the measure of similarity δ^3 used in the NLMeans method. In the first two examples that follow, we take the window size $\eta = 20$ pixels, the size of the patch $\rho = 7$ pixels and $\varepsilon = 5$ pixels. As indicated in assumption (iii), for each fixed patch, we only keep the $m_0 = 10$ best matches. Finally, the selectivity parameter $h = 100$. In the examples on page 148, 149, we set $\eta = 100$, $\rho = 5$, $\varepsilon = 3$, $m_0 = 6$, $h = 100$, $p = 4$. In the last simulation we chose $\eta = 60$, $\rho = 9$, $\varepsilon = 3$, $m_0 = 10$, $h = 100$, $p = 4$.
- The constrained minimization problem (4.3.2) where the weight is computed with the atom-based distance δ^1 introduced in this chapter. We consider the $n_0 = 25$ first atoms where the moment $p = 4$. And as above we take $\varepsilon = 5$ pixels, $h = 100$ and for a fixed patch, we only consider the $m_0 = 10$ best matches. In the examples on page 148, we take $\eta = 100$, $\varepsilon = 3$, $m_0 = 6$, $n_0 = 18$, $h = 100$, $p = 4$. To produce the last tests we set $\eta = 60$, $\varepsilon = 3$, $m_0 = 10$, $n_0 = 18$, $h = 100$, $p = 4$.

We compare all these methods using the standard PSNR. The higher this value is the better the restoration is.

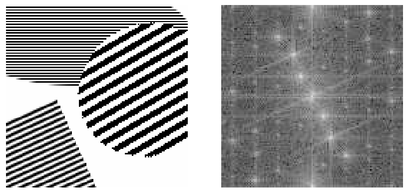


Figure 4.25: 128 × 128 toy example image g and its spectrum,

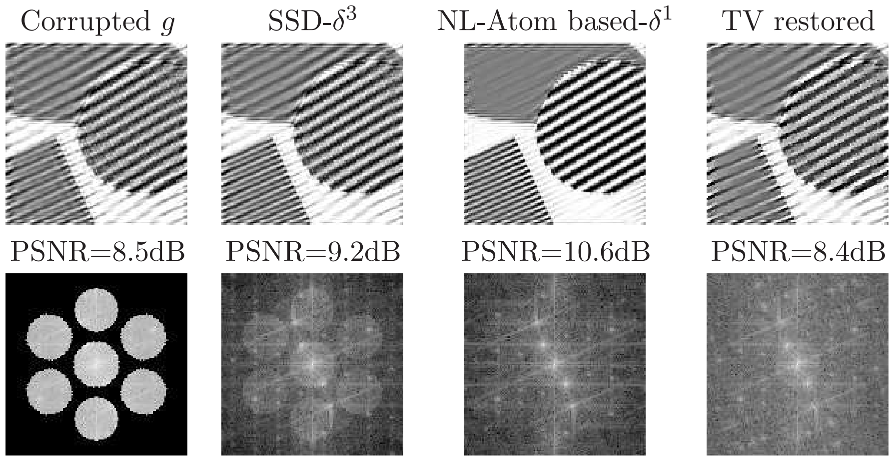


Figure 4.26: Corrupted and restored images and their respective spectra.

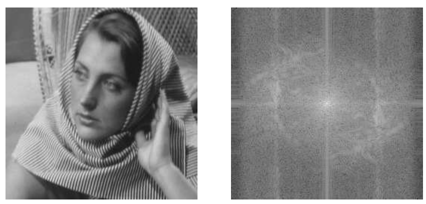


Figure 4.27: 256 × 256 crop of the *Barbara* image g and its spectrum,

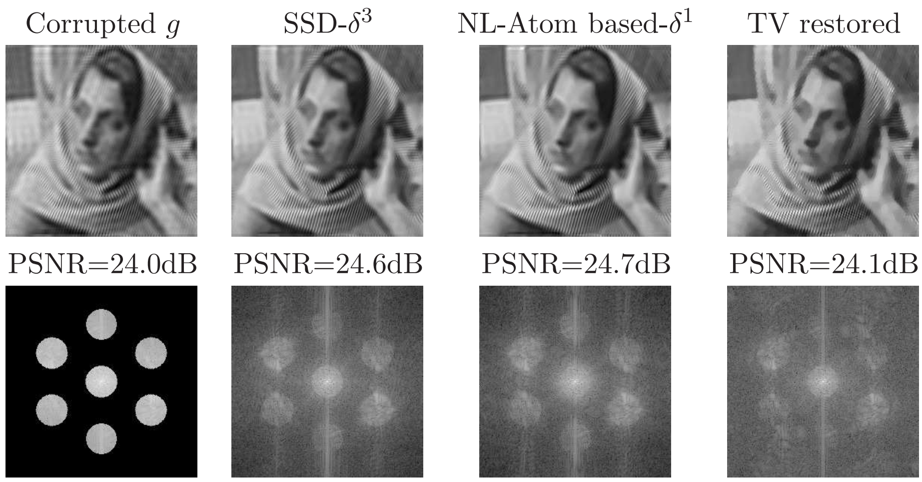


Figure 4.28: Corrupted and restored 256×256 crop of *Barbara* and their respective spectra.

Let us see how these methods perform for the acoustic scattering problem we introduced previously. To do so, we consider the mask introduced in Figure 4.1.

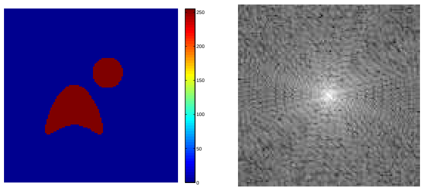


Figure 4.29: 128×128 image of two scatterers and the corresponding spectrum.

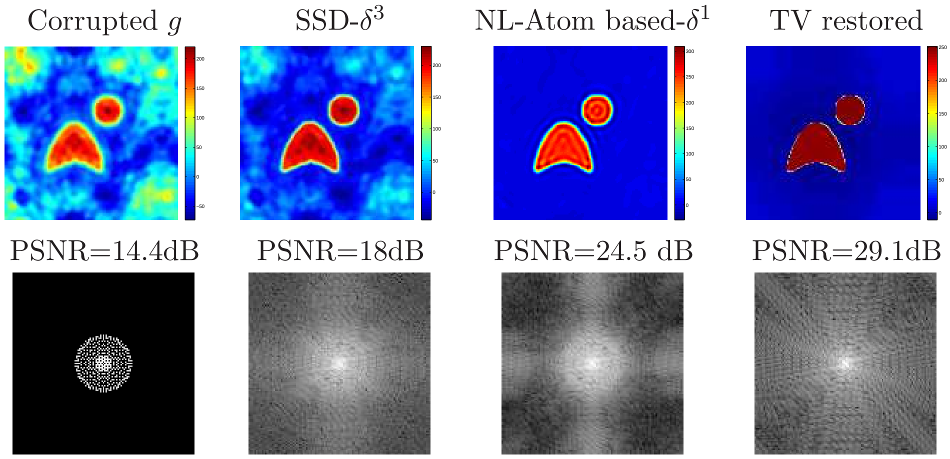


Figure 4.30: Corrupted and restored scatterers and their respective spectra.

In the next simulation, we are going to assume that the Fourier coefficients that we kept are contaminated by a Gaussian noise of magnitude $0.03\|g_0\|_2$.

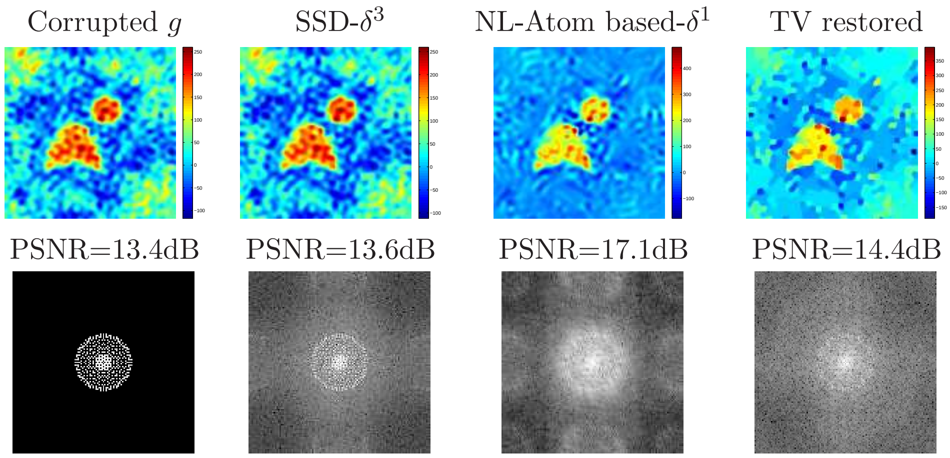


Figure 4.31: Corrupted and restored scatterers and their respective spectra.

Now considering the Born approximation (4.2.1), let us use the data that comes out of the direct problem. In a sense, this amounts to add some noise to the Fourier coefficients. Our method is able to get rid of all the spurious objects.

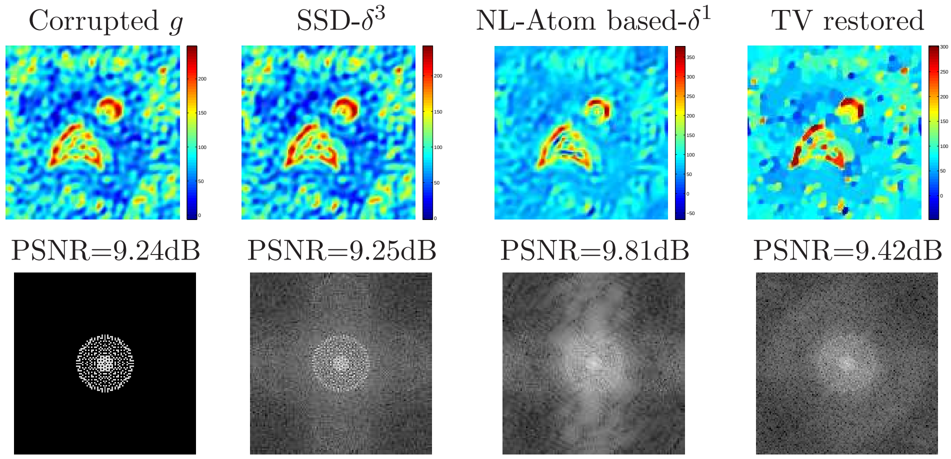


Figure 4.32: Corrupted and restored scatterers and their respective spectra.

In this kind of problems it is usually important to distinguish objects that are very close. Let us consider such a situation:

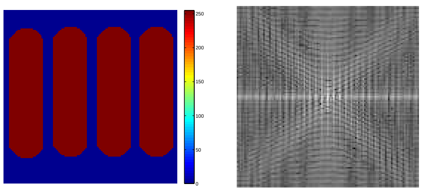


Figure 4.33: 128×128 image of scatterers separated by 6 pixels and its spectrum

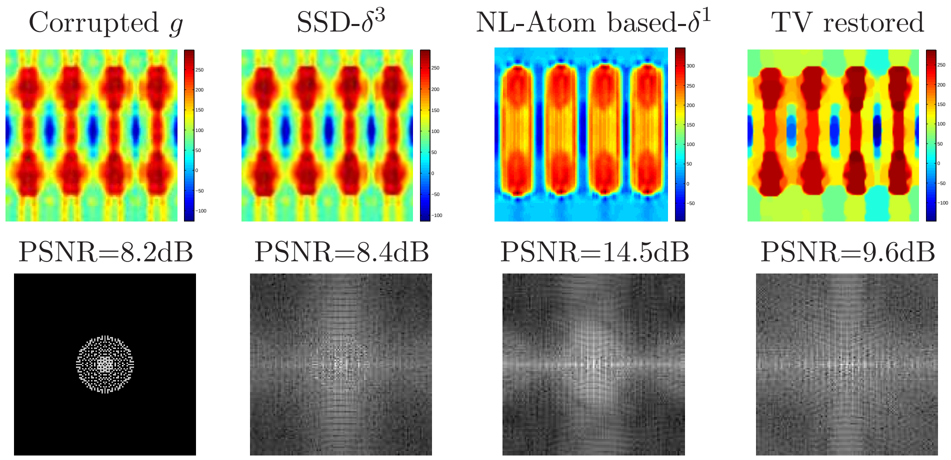
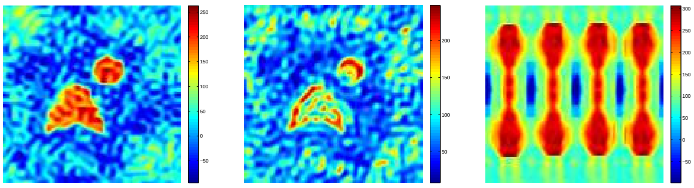


Figure 4.34: Corrupted and restored scatterers and their respective spectra.

In the latter experiment, we were able to distinguish objects that are separated by at least 0.56λ which is quite close to the theoretical limit $\lambda/2$.

Weight recomputation: In [91, 90, 135, 136, 15], it was pointed out that one can get improved results (especially for inpainting problems) by allowing recomputation of the NLMeans weight on the being restored image. Though *really cumbersome*, here are the kind of restoration results one can expect after several recomputations of the weight:

PSNR=12.1dB PSNR=24.9dB PSNR=22.3 dB

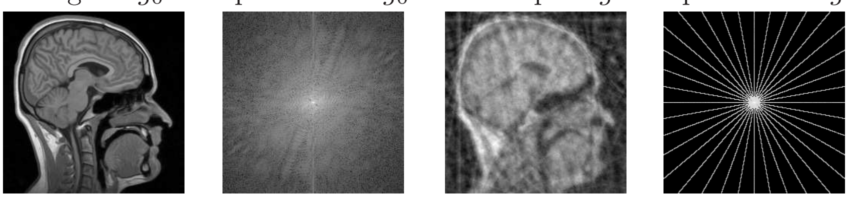


PSNR=14.6dB PSNR=9.27dB PSNR=8.9dB

Figure 4.35: Restored images after many SSD weight recomputations.

This procedure, that does not always improve results over the atom-based method, is not possible for the atom-based distance we introduced since the distance computed on the restored image is exactly the same as the one computed on the corrupted image. Though, our method can be used as an initialization for the classical weight recomputation to improve results further. This is the strategy we adopt in the following tomography problem where the Fourier coefficients got corrupted by a Gaussian noise of magnitude $0.3\|g_0\|_2$:

Original g_0 Spectrum of g_0 Corrupted g Spectrum of g



PSNR=22.4dB

NLMeans - δ^3 NL-Atom - δ^1 δ^1 then one comput. of δ^3 δ^3 recomputed 20 times TV restored

PSNR= 23.8dB PSNR=24.9dB PSNR=25.8dB PSNR=24.8dB PSNR=23.6dB

Figure 4.36: Restoration for a 240 × 240 tomography image.

4.7 Conclusion

The object of this work was to design an adequate regularization for the problem of recovering missing Fourier coefficients. We proposed a Non-Local energy based on an original patchwise similarity measure that is adapted to the missing spectrum. The similarity criterion is invariant under the corruption process so that the distance between two patches of the corrupted image is exactly equal to the one computed on the clean image. We illustrated our method with experiments which showed its efficiency, both in terms of speed and quality of the results, with respect to other common approaches. We showed that the method is practical on synthetic examples which are built upon models of inverse scattering problems, synthetic aperture mirrors for spatial imaging or also medical imaging.

CHAPTER 5

Convex Optimization: Algorithmic Aspects

Contents

5.1 Introduction

In this chapter, we aim to present some algorithms that were used in this thesis and that are important for the minimization of general convex problems. The idea is that the more we know about the structure of the problem the faster the designed algorithms can be. The first algorithms we recall are quite classical and one can find more details in the textbook [130]. For a relatively recent review of the most popular algorithms and their connections we refer to [82]. We shall devote an important section to the very recent Primal-Dual algorithms [159, 82, 59].

Contributions:

- We give a proof of convergence for the Arrow-Hurwicz algorithm with constant stepsize and prove that this convergence is better than $O\left(\frac{1}{n}\right)$.
- We also prove, that with a slight modification of the primal-dual algorithm with varying stepsize of [59], we can obtain a rate of convergence that is beyond $O\left(\frac{1}{n^2}\right)$. This is to our knowledge the best result in this direction.
- Finally, we compare the performance of all these classical algorithms for the ROF denoising problem.

Henceforth, $X = \mathbb{R}^N$ and $Y = \mathbb{R}^M$ for some $N, M \in \mathbb{N}$. They will be equipped with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$.

5.2 Conjugate Gradient

Given a matrix $A : X \to Y$, $b \in Y$ and the quadratic form $Q(x) = \frac{1}{2}\|Ax - b\|^2$. We are interested in the problem

$$\inf_{x \in X} Q(x).$$

Note that here we do not assume that the matrix is symmetric or definite-positive. A might not even be a square matrix. For a comprehensive introduction to this algorithm we refer to [145, 139].

Algorithm 5.1 Conjugate gradient

- **Initialization:** Pick $x^0 \in X$, $r^0 = b - Ax^0$, $z^0 = A^* r^0$, $p^0 = z^0$.
- **Iterations:** For $n \geq 0$, consider the updates:

$$\begin{array}{rcl|rcl} w^n & = & Ap^n, & z^{n+1} & = & A^* r^{n+1}, \\ \alpha^n & = & \|z^n\|^2 / \|w^n\|^2, & \beta^n & = & \|z^{n+1}\|^2 / \|z^n\|^2, \\ x^{n+1} & = & x^n + \alpha^n p^n, & p^{n+1} & = & z^{n+1} + \beta^n p^n. \\ r^{n+1} & = & r^n - \alpha^n w^n, & & & \end{array}$$

It is well known that, theoretically, the iterate x^n reaches a minimizer $\hat{x}$ of Q in a finite number of iterations which is not larger than the size of A that is $N \times M$.

The algorithm can actually be used for solving the system $Ax = b$ for an invertible A. This kind of problem arises in many finite difference or finite elements problems. The idea is to minimize the quadratic form $Q(x) = \frac{1}{2}\|Ax - b\|_2^2$ which is much faster than inverting the matrix A of size $N \times N$ (whose complexity is bounded from below by $O(N^2 \ln(N))$). Minimizing the latter quadratic form allows oneself to solve $Ax = b$ even though A is not invertible, not a square matrix or even in case $b \notin \mathcal{R}(A)$, the range of A.

5.3 General non smooth convex problem

Given a convex set C and some $C^{1,1}$ convex F (with L the Lipschitz constant of ∇F), we aim to solve

$$\inf_{x \in C} F(x).$$

Nesterov proposed in 2005 to construct a minimizing sequence $(y^n)_{n\in\mathbb{N}}$ for this problem in the following way:

Algorithm 5.2 Nesterov's scheme [131]

- **Initialization:** Pick $x^0 \in C$.
- **Iterations:** For $n \geq 0$, perform the updates:

$$\begin{aligned} y^n &= \operatorname*{argmin}_{x\in C} \langle \nabla F(x^n), y - x^n \rangle + \frac{L}{2}\|y - x^k\|^2, \\ z^n &= \operatorname*{argmin}_{x\in C} \sum_{k=0}^{n} \frac{k+1}{2}(f(x^k) + \langle G(x^k), x - x^k\rangle) + \frac{L}{2}\left\|x - x^0\right\|^2, \\ x^{n+1} &= \frac{2}{n+3}z^n + \frac{n+1}{n+3}y^n. \end{aligned}$$

If $C = X$ things can be rewritten as follows:

Algorithm 5.3 Nesterov's scheme [131] when $C = X$

- **Initialization:** Pick $x^0 \in X$.
- **Iterations:** For $n \geq 0$, perform the updates:

$$\left.\begin{aligned} y^n &= x^n - L^{-1}\nabla F(x^n), \\ z^n &= x^0 - \tfrac{1}{2L}\textstyle\sum_{k=0}^{n}(k+1)\nabla F(x^k), \end{aligned}\right| \quad x^{n+1} = \tfrac{2}{n+3}z^n + \tfrac{n+1}{n+3}y^n.$$

If the Lipschitz constant L is not known then one can perform a *linesearch* to find the optimal stepsize (see [130] for more on the subject).

This algorithm is proven to converge:

Theorem 5.3.1. *Given a minimizer $\hat{x}$ one has*

$$F(y^n) - F(\hat{x}) \leq \frac{L\|x_0 - \hat{x}\|}{(n+1)(n+2)}.$$

Thus one has

Corollary 5.3.2. *Let F be uniformly convex and consider a minimizer $\hat{x}$ then*

$$\|y^n - \hat{x}\|^2 \leq \frac{L\|x_0 - \hat{x}\|}{(n+1)(n+2)}.$$

5.4 A splitting algorithm for composite objective

Let F be of class $C^{1,1}$ with L the Lipschitz constant of the gradient ∇F and assume that G is such that its *resolvent operator*, also called *proximal operator*, $(I + \alpha\partial G)^{-1}$ (that we introduced in section 2.6) is easily computed. We recall that this means that it is possible to solve

$$(I + \alpha\partial G)^{-1}(x) = \operatorname*{argmin}_{x' \in X} \frac{1}{2}\|x' - x\|^2 + \alpha G(x').$$

In such a case we shall say that G is *simple* (see also [69] on the subject).

We aim to solve

$$\inf_{x \in X} F(x) + G(x)$$

where the objective is the sum of two functions. As already seen, this kind of problem is quite classical for solving general inverse problems in image processing. The following algorithm was recently proposed for such composite objectives:

Algorithm 5.4 Beck-Teboulle's scheme [21]

- **Initialization:** Pick $x^0 \in X$, $y^1 = x^0$, $t_1 = 1$.
- **Iterations:** For $n \geq 1$, perform the updates:

$$\begin{aligned} x^n &= (I + L^{-1}\partial G)^{-1}(y^n - L^{-1}\nabla F(y^n)), \\ t_{n+1} &= \frac{1 + \sqrt{1 + 4{t_n}^2}}{2}, \\ y^{n+1} &= x^n + (t_n - 1)t_{n+1}^{-1}(x^n - x^{n-1}). \end{aligned}$$

Then the following convergence result is proven in [21]:

Theorem 5.4.1. *Consider a minimizer $\hat{x}$ then one has*

$$(F(x_n) + G(x_n)) - (F(\hat{x}) - G(\hat{x})) \leq \frac{2L\|x_0 - \hat{x}\|^2}{(n+1)^2}.$$

And if G is uniformly convex this implies

Corollary 5.4.2. *Let $\hat{x}$ be a minimizer then*

$$\|x^n - \hat{x}\|_2^2 \leq \frac{2L\|x_0 - \hat{x}\|^2}{(n+1)^2}.$$

A $O(\frac{1}{n^2})$ variant of this algorithm for minimizing composite objective functions has first been introduced by Nesterov in [132], though it is in practice slower than the algorithm of Beck and Teboulle.

Remark 5.4.3. Given a convex C, if we set $G = \iota_C$ the indicator of C, we get back the problem

$$\min_{x \in C} F(x).$$

The proximal operator in this case is simply the projection on the convex C.

5.5 The Primal-Dual framework

In the context of image processing, a Primal-Dual type algorithm was proposed in [159] for the minimization of ROF's functional but was probably first studied in [16]. Given two convex and lower semicontinuous functions $G : X \to [0, +\infty)$, $F : Y \to [0, +\infty)$ and a matrix $A : X \to Y$, the method amounts to write

$$\min_{x \in X} F(Ax) + G(x)$$

as a saddle-point problem

$$\min_{x \in X} \max_{y \in Y} \langle Ax, y \rangle + G(x) - F^*(y). \tag{5.5.1}$$

thanks to the Legendre transformation of F. We assume that there exists at least one solution that we shall denote $(\hat{x}, \hat{y})$. The general idea for solving such a problem is to perform iteratively a gradient descent in the variable x and a gradient ascent in y which gives the following generic algorithm

$$\begin{cases} x^{n+1} & = (I + \tau \partial G)^{-1}(x^n - \tau A^* \bar{y}^n), \\ y^{n+1} & = (I + \sigma \partial F^*)^{-1}(y^n + \sigma A \bar{x}^n), \end{cases}$$

for some positive stepsizes σ, τ and $\bar{x}^n, \bar{y}^n$ that we are going to set in the sequel and that will allow us to consider different algorithms. In [59], the authors consider such a general scheme and prove various results of convergence. Their analysis is inspired by an old paper of Popov [137]. In [82], it is proven that many algorithms in the literature can be casted more or less into this form.

Henceforth, we are going to deal with the case when G is uniformly convex. In other words, we assume there exists $\gamma > 0$ such that for any $x \in \operatorname{dom} \partial G$

$$G(x') - G(x) \geq \langle p, x' - x \rangle + \frac{\gamma}{2}\|x - x'\|^2. \tag{5.5.2}$$

The case when F^* is uniformly convex can be handled in an equivalent way.

For such a G, it is proven in [59, Eq. (37)] that one has the following estimate:

$$\begin{aligned}\frac{\|\hat{y} - y^n\|^2}{\sigma} + \frac{\|\hat{x} - x^n\|^2}{\tau} &\geq 2\gamma\|\hat{x} - x^{n+1}\|^2 \\ &+ \frac{\|\hat{y} - y^{n+1}\|^2}{\sigma} + \frac{\|\hat{x} - x^{n+1}\|^2}{\tau} + \frac{\|y^{n+1} - y^n\|^2}{\sigma} + \frac{\|x^{n+1} - x^n\|^2}{\tau} \\ &+ 2\langle A(x^{n+1} - \bar{x}^n), y^{n+1} - \hat{y}\rangle - 2\langle A(x^{n+1} - \hat{x}), y^{n+1} - \bar{y}^n\rangle. \end{aligned} \tag{5.5.3}$$

5.5.1 Convergence of the Arrow-Hurwicz algorithm

In this section, we are interested in the Arrow-Hurwicz scheme which amounts to choose

$$\bar{x}^n = x^{n+1}, \; \bar{y}^n = y^n,$$

This choice corresponds to the following semi-implicit algorithm:

Algorithm 5.5 Arrow-Hurwicz's scheme

- **Initialization:** Pick $x^0 \in X$, $y^0 \in Y$, σ, τ with $\sigma\|A\|^2 \leq \gamma$.
- **Iterations:** For $n \geq 1$ update as follows:

$$\begin{aligned} x^{n+1} &= (I + \tau\partial G)^{-1}(x^n - \tau A^* y^n), \\ y^{n+1} &= (I + \sigma\partial F^*)^{-1}(y^n + \sigma A x^{n+1}). \end{aligned}$$

A proof of convergence for the ROF problem is given in [82] but the conditions on the stepsizes are quite restrictive and result in a very slow algorithm. In [59], the authors prove a $O(1/\sqrt{n})$ convergence result for this algorithm in case G is merely convex but they suppose in addition that $\operatorname{dom}(F^*)$ is bounded (which is true for the ROF problem). In the following we propose a really simple proof of convergence in case G is uniformly convex.

First, let us denote

$$\Delta_n = \frac{\|\hat{y} - y^n\|^2}{\sigma} + \frac{\|\hat{x} - x^n\|^2}{\tau}.$$

From (5.5.3) we then get

$$\Delta_n \geq \Delta_{n+1} + 2\gamma\|\hat{x} - x^{n+1}\|^2 + \frac{\|y^{n+1} - y^n\|^2}{\sigma} + \frac{\|x^{n+1} - x^n\|^2}{\tau}$$
$$- \gamma\|\hat{x} - x^{n+1}\|^2 - \frac{\|A\|^2}{\gamma}\|y^{n+1} - y^n\|^2.$$

Therefore, since $\sigma \leq \frac{\gamma}{\|A\|^2}$,

$$\Delta_n \geq \Delta_{n+1} + \gamma\|\hat{x} - x^{n+1}\|^2. \tag{5.5.4}$$

Summing from $n = 0$ to $+\infty$, it follows

$$\gamma \sum_n \|\hat{x} - x^n\|^2 \leq \Delta_0.$$

So far we have proven

Theorem 5.5.1. *Let $\tau, \sigma > 0$ such that $\sigma \leq \frac{\gamma}{\|A\|^2}$. Then the sequence $(x^n)_{n\in\mathbb{N}}$ converges to $\hat{x}$ and the error $\|\hat{x} - x^n\|^2$ is summable.*

Remark 5.5.2.(i) This means that the convergence rate is better that $O\left(\frac{1}{n}\right)$ though one cannot conclude that $x_n = o\left(\frac{1}{n}\right)$ since $(x_n)_{n\in\mathbb{N}}$ is not necessarily a decreasing sequence.

(ii) As was done in [59], from this point it is possible to prove convergence of $(y^n)_{n\in\mathbb{N}}$ to some $\hat{y}' \in Y$ that might be different from $\hat{y}$ but such that $(\hat{x}, \hat{y}')$ is still a saddle point of (5.5.1).

(iii) In [94], the Arrow-Hurwicz algorithm is studied in the continuous framework. The author proves an analogous of (5.5.4) (see [94, Eq. (9)]) which as in the discrete setting implies convergence.

5.5.2 Dependency on γ of the adaptive stepsize algorithm

Let us take

$$\bar{x}^n = x^n + \theta_{n-1}(x^n - x^{n-1}),\ \bar{y}^n = y^{n+1},$$

for some positive θ_{n-1} to be set later on. Then, making σ and τ depend on the iterations with $\sigma_0, \tau_0 > 0$ such that $\sigma_0\tau_0\|A\|^2 \leq 1$, one can consider the scheme

Algorithm 5.6 Primal Dual with adaptive stepsize of [59]

- **Initialization:** Pick $x^0 = \bar{x}^0 \in X$, $y^0 \in Y$, $\sigma_0, \tau_0 \geq 0$ with $\sigma_0\tau_0\|A\|^2 \leq 1$.
- **Iterations:** For $n \geq 1$, consider the updates:

$$\begin{aligned}
y^{n+1} &= (I + \sigma_n \partial F^*)^{-1}(y^n + \sigma_n A\bar{x}^n),\\
x^{n+1} &= (I + \tau_n \partial G)^{-1}(x^n - \tau_n A^* y^{n+1}),\\
\theta_n &= 1/\sqrt{1 + 2\gamma\tau_n},\ \tau_{n+1} = \theta_n\tau_n,\ \sigma_{n+1} = \sigma_n/\theta_n,\\
\bar{x}^{n+1} &= x^{n+1} + \theta_n(x^{n+1} - x^n).
\end{aligned}$$

A. Chambolle and T. Pock prove that whenever $\sigma_0\tau_0\|A\|^2 = 1$ one has

$$\|\hat{x} - x^n\|^2 \leq \frac{1}{n^2}\left(\eta + \frac{\|A\|^2}{\gamma^2}\|\hat{y} - y^0\|^2\right)$$

for some small η. In other words, the previous scheme converges as $O(\frac{1}{n^2})$. This is to be compared with the aforementioned algorithms.

Though, we observed that if we pick the optimal $\gamma = \gamma_0$ in (5.5.2) (that is, for a smooth G, the best constant such that $D^2G(x) \geq \gamma$) the algorithm may perform very poorly. To see this, we minimize the discretized ROF functional where we set $\lambda = 10$ and g is a noisy 64×64 image. We set $\tau_0 = \frac{100}{L^2}, \sigma_0 = \frac{1}{\tau_0 L^2}$ with $L = \sqrt{8}$. Then we compare the rate of convergence of $(x_n)_{n\in\mathbb{N}}$ given different γ:

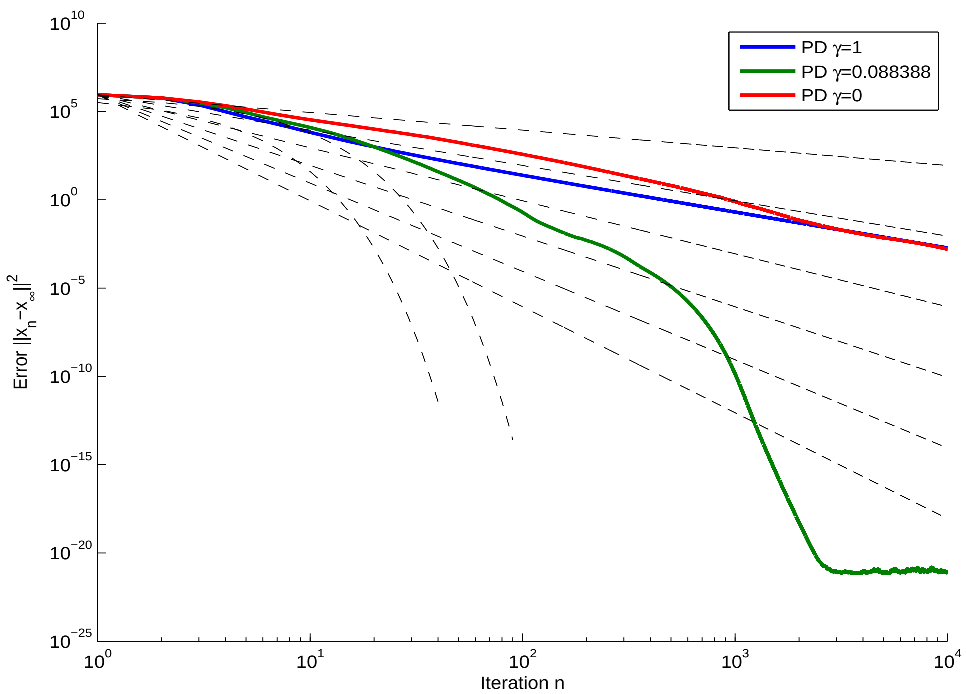


Figure 5.1: Error $\|x^n - \hat{x}\|^2$ for different values of the convexity parameter γ

Dashed lines are used to represent respectively rates of convergence $O\left(\frac{1}{n^k}\right)$ for $k = 1, ..., 6$ and also some linear convergence rates. We can observe that, for the optimal value γ_0 (for ROF, $\gamma_0 = 1$), the algorithm converges as $O(\frac{1}{n^2})$ but not faster as it seems to be for smaller values of γ. Note that $\gamma = 0$ actually corresponds to a scheme where the stepsizes remain constant. It appears that the choice of the uniform convexity parameter γ is therefore critical.

5.5.3 A result of convergence beyond $O(\frac{1}{n^2})$

The observation of the previous section suggests a modification of the algorithm that allows us to improve the theoretical rate of convergence. Let us change slightly the iteration of θ_n which is the only place where γ appears:

Algorithm 5.7 Proposed variant of Primal Dual

- **Initialization:** Pick $x^0 = \bar{x}^0 \in X$, $y^0 \in Y$, $\sigma_0, \tau_0 \geq 0$ with $\sigma_0\tau_0\|A\|^2 \leq 1$.
- **Iterations:** For $n \geq 1$, consider the following updates:

$$\begin{aligned}
y^{n+1} &= (I + \sigma_n \partial F^*)^{-1}(y^n + \sigma_n A\bar{x}^n), \\
x^{n+1} &= (I + \tau_n \partial G)^{-1}(x^n - \tau_n A^* y^{n+1}), \\
\theta_n &= 1/\sqrt{1 + \gamma\tau_n}, \ \tau_{n+1} = \theta_n \tau_n, \ \sigma_{n+1} = \sigma_n/\theta_n, \\
\bar{x}^{n+1} &= x^{n+1} + \theta_n(x^{n+1} - x^n).
\end{aligned}$$

We are going to prove that this algorithm converges by adapting the argument of Chambolle-Pock. We get from (5.5.3)

$$\begin{aligned}
\frac{\|\hat{y} - y^n\|^2}{\sigma_n} + \frac{\|\hat{x} - x^n\|^2}{\tau_n} \geq\ & 2\gamma\|\hat{x} - x^{n+1}\|^2 \\
& + \frac{\|\hat{y} - y^{n+1}\|^2}{\sigma_n} + \frac{\|\hat{x} - x^{n+1}\|^2}{\tau_n} + \frac{\|y^{n+1} - y^n\|^2}{\sigma_n} + \frac{\|x^{n+1} - x^n\|^2}{\tau_n} \\
& + 2\langle A(x^{n+1} - x^n), y^{n+1} - \hat{y}\rangle - 2\theta_{n-1}\langle A(x^n - x^{n-1}), y^n - \hat{y}\rangle \\
& - 2\theta_{n-1}\|A\|\|x^n - x^{n-1}\|\|y^{n+1} - y^n\| \qquad (5.5.5)
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\|\hat{y} - y^n\|^2}{\sigma_n} + \frac{\|\hat{x} - x^n\|^2}{\tau_n} \geq\ & \theta_n^2\|\hat{x} - x^{n+1}\|^2 + \frac{\|\hat{y} - y^{n+1}\|^2}{\sigma_n} \\
& + \frac{\|x^{n+1} - x^n\|^2}{\tau_n} - \sigma_n\theta_{n-1}^2\|A\|^2\|x^n - x^{n-1}\|^2 \\
& + 2\langle A(x^{n+1} - x^n), y^{n+1} - \hat{y}\rangle - 2\theta_{n-1}\langle A(x^n - x^{n-1}), y^n - \hat{y}\rangle \\
& + \gamma\|\hat{x} - x^{n+1}\|^2
\end{aligned}$$

where we used

$$2\theta_{n-1}\|A\|\|x^n - x^{n-1}\|\|y^{n+1} - y^n\| \le \sigma_n \theta_{n-1}^2 \|A\|^2 \|x^n - x^{n-1}\|^2 + \frac{\|y^{n+1} - y^n\|^2}{\sigma_n}.$$

As previously, denoting

$$\Delta_n = \frac{\|\hat{y} - y^n\|^2}{\sigma_n} + \frac{\|\hat{x} - x^n\|^2}{\tau_n},$$

and remarking that by definition

$$\theta_n^2 \frac{\tau_{n+1}}{\tau_n} = \frac{\sigma_{n+1}}{\sigma_n} = \frac{\tau_n}{\tau_{n+1}}$$

it follows

$$\begin{aligned}\frac{\Delta_n}{\tau_n} \ge \frac{\Delta_{n+1}}{\tau_{n+1}} &+ \frac{\|x^{n+1} - x^n\|^2}{\tau_n^2} - \frac{\|x^n - x^{n-1}\|^2}{\tau_{n-1}^2} \\ &+ \frac{2}{\tau_n}\langle A(x^{n+1} - x^n), y^{n+1} - \hat{y}\rangle - \frac{2}{\tau_{n-1}}\langle A(x^n - x^{n-1}), y^n - \hat{y}\rangle \\ &+ \frac{\gamma}{\tau_n}\|\hat{x} - x^{n+1}\|^2\end{aligned}$$

since we have

$$\|A\|^2 \sigma_n \tau_n = \|A\|^2 \sigma_0 \tau_0 \le 1.$$

Then summing we can telescope terms so we get in the end

$$\gamma \sum_n \frac{\|\hat{x} - x^{n+1}\|^2}{\tau_n} < +\infty.$$

Though, since we know from the definition of τ_n (see [59, Corollary 1]) that

$$\tau_n \sim \frac{2}{\gamma n}$$

when $n \to +\infty$, we have proven so far

Theorem 5.5.3. *Let $\tau_0, \sigma_0 > 0$ such that $\sigma_0 \tau_0 \|A\|^2 \le 1$ then the sequence $(x^n)_{n\in\mathbb{N}}$ converges to $\hat{x}$ and*

$$\sum_n n\|\hat{x} - x^n\|^2 < +\infty. \tag{5.5.6}$$

To our knowledge, the resulting complexity for this class of problems is the best theoretical rate of convergence in the literature that is to say beyond the $O\left(\frac{1}{n^2}\right)$ estimates of [131, 132, 21, 59] and is beyond the theoretical convergence rate of first order gradient methods for convex problems (see [130, Theorem 2.1.7]). This is not surprising since we have additional assumptions on the structure of our problem. In case both G and F^* are uniformly convex, it is proven in [59] that one can actually achieve a linear convergence rate.

Remark 5.5.4. If one assumes $\sigma_0\tau_0\|A\|^2 < 1$, that is to say there exists $0 < \varepsilon < 1$ such that $\sigma_0\tau_0\|A\|^2 = 1 - \varepsilon$, it follows from (5.5.5) that

$$\begin{aligned}\frac{\Delta_n}{\tau_n} &\geq \frac{\Delta_{n+1}}{\tau_{n+1}} + (1-\varepsilon)\left(\frac{\|x^{n+1}-x^n\|^2}{\tau_n^2} - \frac{\|x^n - x^{n-1}\|^2}{\tau_{n-1}^2}\right)\\ &+\frac{2}{\tau_n}\langle A(x^{n+1}-x^n), y^{n+1}-\hat{y}\rangle - \frac{2}{\tau_{n-1}}\langle A(x^n - x^{n-1}), y^n - \hat{y}\rangle\\ &+\frac{\gamma}{\tau_n}\|\hat{x} - x^{n+1}\|^2 + \varepsilon\frac{\|x^{n+1}-x^n\|^2}{\tau_n^2}\end{aligned}$$

which implies in addition to (5.5.6) that

$$\sum_n n^2\|x^{n+1}-x^n\|^2 < +\infty.$$

Said otherwise, the sequence $(x_n)_{n\in\mathbb{N}}$ does not oscillate too much.

5.6 Comparison for the TV denoising problem

In this section, we compare the Primal-Dual algorithm with some popular methods to see how well they either minimize the primal formulation of the denoising problem, namely

$$\min_u \lambda \int_\Omega |Du| + \frac{1}{2}\|u-g\|_2^2 \tag{5.6.1}$$

or equivalently minimize the smooth but constrained dual problem (see [54])

$$\min_{\|z\|_\infty \leq \lambda} \|\operatorname{div}(z) + g\|_2^2. \tag{5.6.2}$$

In the dual case, if z_λ is a minimizer, the restored image is given by $u_\lambda = \operatorname{div}(z_\lambda) + g$.

We carry out some tests with a fixed regularization parameter $\lambda = 10$ and we consider a 512×512 pixels image $g = g_0 + n$ where n is a Gaussian noise.

We are going to consider the following algorithms:

- **Zhu-Chan:** This is an implementation of the Arrow-Hurwicz algorithm according to the update of the stepsizes described in [159].

- **Adaptive Arrow-Hurwicz:** We consider the Arrow-Hurwicz algorithm where the stepsizes are updated according to Algorithm 5.7. We take $\tau_0 = \frac{20}{L}, \sigma_0 = \frac{1}{\tau_0 L^2}$ with $L = \sqrt{8}$ and $\gamma = 0.06$.

- **Nesterov Dual:** Nesterov's algorithm [131] on the dual problem (5.6.2).

- **Beck-Teboulle Dual:** The algorithm proposed in [21] for the dual problem (5.6.2).

- **Proj-Grad Dual:** Projected gradient on the dual problem which corresponds to the Arrow-Hurwicz algorithm with the choice $\sigma = +\infty$ and $\tau = \frac{1}{4}$.

- **Chambolle Dual:** Chambolle's fixed point algorithm with $\tau = \frac{1}{4}$ proposed in [54] for the dual problem (5.6.2) and which is proven to converge for $\tau \leq \frac{1}{8}$.

- **Adaptive Primal-Dual:** Algorithm 5.7 with $\tau_0 = \frac{40}{L}$, $\sigma_0 = \frac{1}{\tau_0 L^2}$ where $L = \sqrt{8}$ and $\gamma = 0.06$.

In the following figures, we plotted some performance tests for all these algorithms. An approximation of $\hat{x} = u_\lambda$ has been first computed by doing 10^5 iterations of the algorithm of Zhu and Chan.

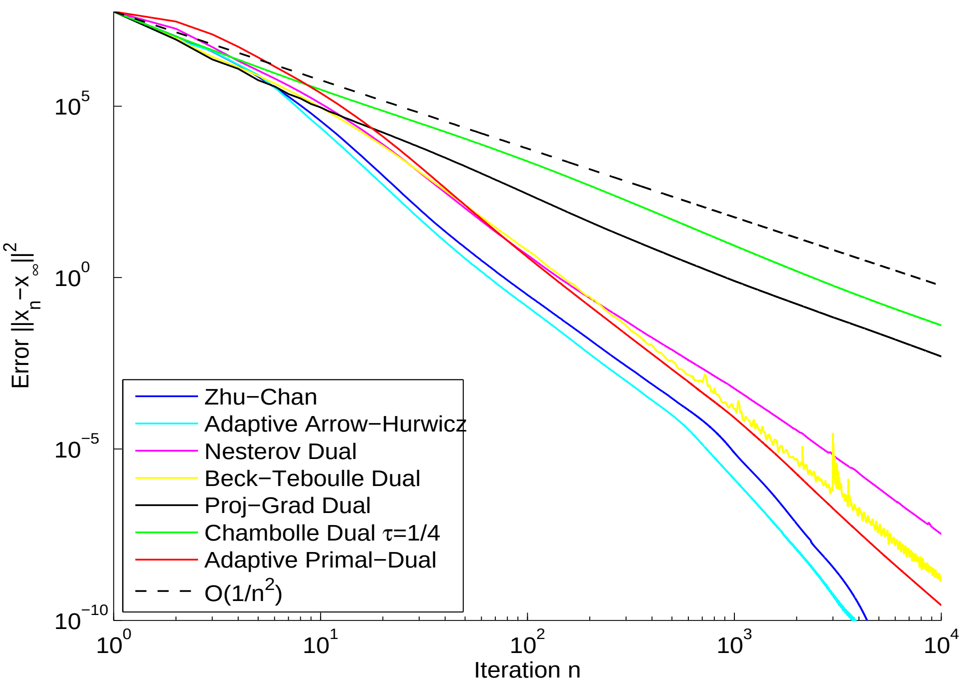


Figure 5.2: Minimizer error $\|x^n - \hat{x}\|^2$ for the considered algorithms

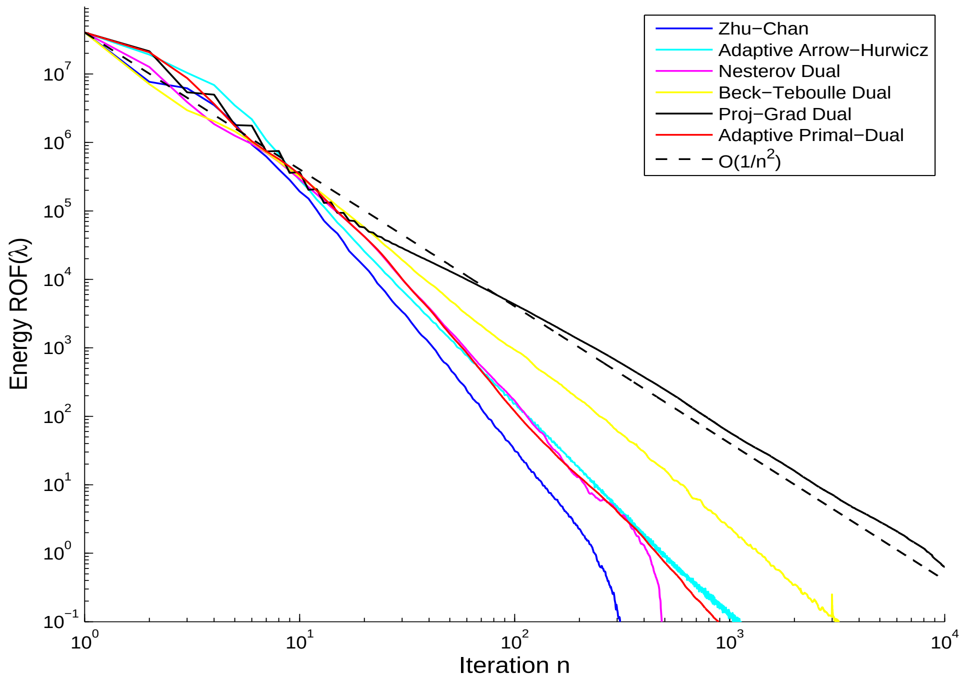


Figure 5.3: Energy error $\mathcal{E}_\lambda(x^n) - \mathcal{E}_\lambda(\hat{x})$ for the considered algorithms

The latter tests suggest that almost all these algorithms perform much better than $O(\frac{1}{n^2})$ which is the main idea of Theorem 5.5.3.

The previous performance tests matter if one is interested in the accuracy of the calculation. However, if one is simply interested in the visual quality of the enhanced image, one should preferably focus on the evolution of the PSNR:

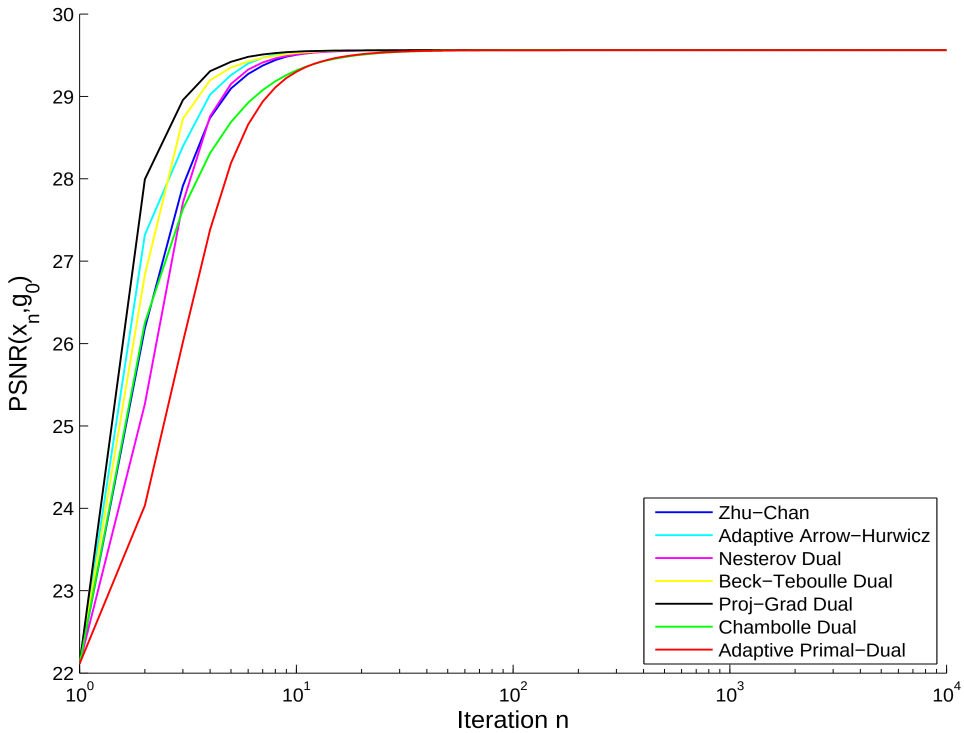


Figure 5.4: PSNR(x^n, g_0) for the considered algorithms

Overall, one can see that in much less than 100 iterations all the work is done and surprisingly the algorithm that performs best is merely the projected gradient on the dual problem.

5.7 Conclusion and perspective

In this chapter, we recalled some recent advances in convex optimization and we examined convergence and complexity of the recently developed Primal-Dual algorithms. Some questions remain unsettled. Indeed, it would be interesting to

- Prove that the dual variable converges for the adaptive Primal-Dual algorithm.
- Devise the optimal uniform convexity parameter γ that gives the best rate of convergence which is in practice beyond $o\left(\frac{1}{n^2}\right)$.

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List of Notations

∇u	Gradient of u
$\operatorname{div} u$	Divergence of u
Du	Derivative of u in the sense of distributions
Δu	Laplacian of u in the sense of distributions
Ω	Open set in $\mathbb{R}^N$
$A \subset\subset \Omega$	A is relatively compact in Ω
$\mathcal{L}^N$	N-dimensional Lebesgue measure
$\mathcal{H}^N$	N-dimensional Hausdorff measure
$\|E\|$	Lebesgue measure of the set E
$\|x\|$	Euclidian norm of vector $x \in \mathbb{R}^N$
$\|\mu\|$	Total variation measure of measure μ
$\operatorname{supp}(\mu)$	Support of measure μ
χ_E	Characteristic function of the set E
$B(x,r)$	Ball of $\mathbb{R}^N$ centered at x and of radius r
$S(x,r)$	Sphere of $\mathbb{R}^N$ centered at x and of radius r
$\mathbb{S}^{N-1}$	Unit sphere in $\mathbb{R}^N$
ω_N	Lebesgue measure of the unit ball in $\mathbb{R}^N$
$\mathcal{D}'(\Omega)$	Space of distributions on Ω

List of Figures

Résumé

Les problèmes inverses consistent à retrouver une donnée qui a été transformée ou perturbée. Ils nécessitent une régularisation puisque mal posés.

En traitement d'images, la variation totale en tant qu'outil de régularisation a l'avantage de préserver les discontinuités tout en créant des zones lisses, résultats établis dans cette thèse dans un cadre continu et pour des énergies générales.

En outre, nous proposons et étudions une variante de la variation totale. Nous établissons une formulation duale qui nous permet de démontrer que cette variante coïncide avec la variation totale sur des ensembles de périmètre fini.

Ces dernières années les méthodes non-locales exploitant les auto-similarités dans les images ont connu un succès particulier. Nous adaptons cette approche au problème de complétion de spectre pour des problèmes inverses généraux.

La dernière partie est consacrée aux aspects algorithmiques inhérents à l'optimisation des énergies convexes considérées. Nous étudions la convergence et la complexité d'une famille récente d'algorithmes dits Primal-Dual.

Abstract

Inverse problems are to recover the data that has been processed or corrupted. Since they are ill-posed they require a regularization.

In image processing, the total variation as a regularization tool has the advantage of preserving the discontinuities while creating smooth regions. These results are established in this thesis in a continuous setting for general energies.

In addition, we propose and examine a variant of the total variation. We establish a dual formulation that allows us to prove that this variant coincides with the total variation for sets of finite perimeter.

Nowadays, non-local methods exploiting the self-similarities of images is particularly successful. We adapt this approach to the problem of spectrum completion, which has applications for general inverse problems.

The final part is devoted to the algorithmic aspects inherent to the optimization of the convex energies we considered. We study the convergence and the complexity of the recently developed Primal-Dual algorithms.